MATH 412, FALL 2012 - HOMEWORK 13

WARMUP PROBLEMS: Section 6.3 #1, 3, 4, 16. Section 7.1 #1, 2, 4, 5, 6. Do not write these up!

EXTRA PROBLEMS: Section 6.3: #5, 6, 10, 11, 15, 17, 21, 24, 26, 30. Section 7.1: #9, 10, 11, 12, 14, 17, 18, 19, 22, 26. Do not write these up!

WRITTEN PROBLEMS: Do five of the following six. Due Wednesday, December 5.

1. **Short proof of the Five Color Theorem.**
   a) Let \( v \) be a 5-vertex in a plane graph \( G \). Let \( x \) and \( y \) be nonadjacent neighbors of \( v \), and let \( G' \) be the graph obtained from \( G \) by contracting the edges \( vx \) and \( vy \). Prove that if \( G' \) is 5-colorable, then \( G \) is 5-colorable.
   b) Use part (a) to give a short inductive proof of the Five Color Theorem.

2. Without using the Four Color Theorem, prove that every outerplanar graph is 3-colorable. Apply this to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with \( n \) sides, then it is possible to place \( \lfloor n/3 \rfloor \) guards such that every point of the interior is visible to some guard. For \( n \geq 3 \), construct a polygon that requires \( \lceil n/3 \rceil \) guards.

3. The **thickness** of a graph \( G \) is the minimum number of planar graphs needed to partition \( E(G) \).
   a) Prove that if \( G \) has thickness 2, then \( \chi(G) \leq 12 \).
   b) For \( r \) even and \( s \) greater than \( (r - 2)^2/2 \), prove that the thickness of \( K_{r,s} \) is \( r/2 \).

4. Suppose that \( m \) and \( n \) are odd. Prove that in all drawings of \( K_{m,n} \), the parity of the number of pairs of nonincident edges that cross an odd number of times is the same. Conclude that \( \nu(K_{m,n}) \) is odd when \( m - 3 \) and \( n - 3 \) are divisible by 4 and even otherwise.

5. Use Tutte’s 1-factor Theorem to prove that every connected line graph of even order has a perfect matching. Conclude from this that every simple connected graph of even size decomposes into paths of length 2. (Comment: Exercise 3.3.23 shows that every connected claw-free graph has a perfect matching; that stronger result is more difficult than this.)

6. **Density conditions for \( \chi'(G) > \Delta(G) \).**
   a) Prove that if \( n(G) = 2m + 1 \) and \( e(G) > m \cdot \Delta(G) \), then \( \chi'(G) > \Delta(G) \).
   b) Prove that if \( G \) is obtained from a \( k \)-regular graph with \( 2m + 1 \) vertices by deleting fewer than \( k/2 \) edges, then \( \chi'(G) > \Delta(G) \).
   c) Prove that if \( G \) is obtained by subdividing an edge of a regular graph with \( 2m \) vertices and degree at least 2, then \( \chi'(G) > \Delta(G) \).