1. **Turán’s proof of Turán’s Theorem**, including uniqueness.
   a) Prove that a maximal simple graph having no $r+1$-clique has an $r$-clique.
   b) Prove that $e(T_{n,r}) = \binom{r}{2} + (n - r)(r - 1) + e(T_{n-r,r})$.
   c) Use parts (a) and (b) to prove Turán’s Theorem by induction on $n$, including the characterization of graphs achieving the bound.

2. **Partial analogue of Turán’s Theorem for $K_{2,m}$**.
   a) Let $G$ be an $n$-vertex graph such that $\sum_{v \in V(G)} \binom{d(v)}{2} > (m - 1)\binom{n}{2}$. Prove that $G$ contains $K_{2,m}$. (Hint: View $K_{2,m}$ as two vertices with $m$ common neighbors.)
   b) Prove that $\sum_{v \in V(G)} \binom{d(v)}{2} \geq e(2e/n - 1)$, where $G$ has $e$ edges.
   c) Use parts (a) and (b) to prove that a graph with more than $\frac{1}{2}(m-1)^{1/2}n^{3/2} + n/4$ edges contains $K_{2,m}$.
   d) Application: Given $n$ points in the plane, prove that the distance is exactly 1 for at most $\frac{1}{\sqrt{2}}n^{3/2} + n/4$ pairs.

3. Let $G_1 = K_1$. For $k > 1$, construct $G_k$ as follows. To the disjoint union $G_1 + \cdots + G_{k-1}$, and add an independent set $T$ of size $\prod_{i=1}^{k-1} n(G_i)$. For each choice of $(v_1, \ldots, v_{k-1})$ in $V(G_1) \times \cdots \times V(G_{k-1})$, let one vertex of $T$ have neighborhood $\{v_1, \ldots, v_{k-1}\}$. (In the sketch of $G_4$ on the left below, neighbors are shown for only two elements of $T$.)
   a) Prove that $\omega(G_k) = 2$ and $\chi(G_k) = k$.
   b) Prove that $G_k$ is $k$-critical.

4. **The Hajós construction**.
   a) For $k \geq 3$, let $G$ and $H$ be $k$-critical graphs sharing only vertex $v$, with $vu \in E(G)$ and $vw \in E(H)$. Let $F$ be the graph formed from $G \cup H$ by deleting the edges $vu$ and $vw$ and adding the edge $uw$ (shown on the right above). Prove that $F$ is $k$-critical.
   b) For all $k \geq 3$, use part (a) to obtain a $k$-critical graph other than $K_k$.
   c) For all $n \geq 4$ except $n = 5$, construct a 4-critical graph with $n$ vertices. (Hint: Part (a) uses the properties of $k$-critical graphs for $G$ and $H$. In part (c), one can give explicit examples with orders 4, 6, 8 and then apply part (a).)

5. Let $G$ be a claw-free graph (no induced $K_{1,3}$).
   a) Prove that the subgraph induced by the union of any two color classes in a proper coloring of $G$ consists of paths and even cycles.
   b) Prove that if $G$ has a proper coloring using exactly $k$ colors, then $G$ has a proper $k$-coloring where the color classes differ in size by at most one.

6. Let $m = k(k+1)/2$. Prove that $K_{m,m-1}$ has no $K_{2k}$-subdivision.