1. For a connected graph \( G \) with at least three vertices, prove that the following statements are equivalent (use of Menger’s Theorem is permitted).
   A) \( G \) is 2-edge-connected.
   B) Every edge of \( G \) appears in a cycle.
   C) \( G \) has a closed trail containing any specified pair of edges.
   D) \( G \) has a closed trail containing any specified pair of vertices.

2. Let \( v \) be a vertex of a 2-connected graph \( G \). Prove that \( v \) has a neighbor \( u \) such that \( G - u - v \) is connected.

3. Let \( G \) be a graph without isolated vertices. Prove that if \( G \) has no even cycles, then every block of \( G \) is an edge or an odd cycle.

4. Suppose that \( \kappa(G) = k \) and \( \text{diam } G = d \). Prove that \( n(G) \geq k(d - 1) + 2 \) and \( \alpha(G) \geq \lceil (1 + d)/2 \rceil \). For each \( k \geq 1 \) and \( d \geq 2 \), construct a graph with connectivity \( k \) and diameter \( d \) for which equality holds in both bounds.

5. A vertex \( k \)-split of a graph \( G \) is a graph \( H \) obtained from \( G \) by replacing one vertex \( x \in V(G) \) by two adjacent vertices \( x_1, x_2 \) such that \( d_H(x_i) \geq k \) and that \( N_H(x_1) \cup N_H(x_2) = N_G(x) \cup \{x_1, x_2\} \).
   a) Prove that every vertex \( k \)-split of a \( k \)-connected graph is \( k \)-connected.
   b) Conclude that any graph obtained from a “wheel” \( W_n = K_1 \vee C_{n-1} \) (Definition 3.3.6) by a sequence of edge additions and vertex 3-splits on vertices of degree at least 4 is 3-connected. (Comment: Tutte [1961b] proved also that every 3-connected graph arises in this way. The characterization does not extend easily for \( k > 3 \).)

6. Given a graph \( G \), let \( D \) be the digraph obtained by replacing each edge with two oppositely-directed edges having the same endpoints (thus \( D \) is the symmetric digraph with underlying graph \( G \)). Assume that for all \( x, y \in V(D) \) both \( \kappa'_D(x, y) = \lambda'_D(x, y) \) and \( \kappa_D(x, y) = \lambda_D(x, y) \) hold, the latter applying only when \( x \nleftrightarrow y \). Use this hypothesis to prove that also \( \kappa'_G(x, y) = \lambda'_G(x, y) \) and \( \kappa_G(x, y) = \lambda_G(x, y) \), the latter for \( x \leftrightarrow y \).