1. Prim’s Algorithm grows a spanning tree from a given vertex of a connected weighted graph \( G \), iteratively adding the cheapest edge from a vertex already reached to a vertex not yet reached, finishing when all the vertices of \( G \) have been reached. (Ties are broken arbitrarily.) Prove that Prim’s Algorithm produces a minimum-weight spanning tree of \( G \).

2. Let \( T \) be a minimum-weight spanning tree in \( G \), and let \( T' \) be another spanning tree in \( G \). Prove that \( T' \) can be transformed into \( T \) by a list of steps that exchange one edge of \( T' \) for one edge of \( T \), such that the edge set is always a spanning tree and the total weight never increases.

3. Prove that the following algorithm correctly finds the diameter of a tree. First, run BFS from an arbitrary vertex \( w \) to find a vertex \( u \) at maximum distance from \( w \). Next, run BFS from \( u \) to reach a vertex \( v \) at maximum distance from \( u \). Report \( \text{diam} \, T = d(u,v) \).

4. Two people play a game on a graph \( G \), alternately picking vertices. Player 1 starts at any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player) and not used before. Thus together they follow a path. The last player who moves wins. Prove that the second player has a winning strategy if \( G \) has a perfect matching, and otherwise the first player has a winning strategy. (Hint: Be careful about the second part!)

5. Use the König–Egerváry Theorem to prove that every bipartite graph \( G \) has a matching of size at least \( e(G)/\Delta(G) \). Use this to conclude that every subgraph of \( K_{n,n} \) with more than \( (k-1)n \) edges has a matching of size at least \( k \).

6. In an \( X,Y \)-bigraph \( G \), the deficiency of a set \( S \) is \( \text{def}(S) = |S| - |N(S)| \); note that \( \text{def}(\emptyset) = 0 \). Prove that \( \alpha'(G) = |X| - \max_{S \subseteq X} \text{def}(S) \). (Hint: Form a bipartite graph \( G' \) such that \( G' \) has a matching that saturates \( X \) if and only if \( G \) has a matching of the desired size, and prove that \( G' \) satisfies Hall’s Condition.)