1. Use Kuratowskis Theorem to prove that $G$ is outerplanar if and only if it has no subgraph that is a subdivision of $K_4$ or $K_{2,3}$. (Hint: To apply Kuratowskis Theorem, consider an appropriate modification of $G$. This is much easier than trying to mimic a proof of Kuratowskis Theorem.)

2. Wagner [1937] proved that a graph $G$ is planar if and only if neither $K_5$ nor $K_{3,3}$ can be obtained from $G$ by performing deletions and/or edge-contractions.
   a) Show that deletion and contraction of edges (and deletion of vertices) preserve planarity. Conclude from this that Wagners condition is necessary.
   b) Use Kuratowskis Theorem to prove that Wagners condition is sufficient.

3. The square of a graph $G$ is the graph $G^2$ where $V(G^2) = V(G)$ and a pair $uv$ is an edge of $G^2$ if and only if $d_G(u,v) \in \{1, 2\}$. Prove that the square of $C_n$ is planar if and only if $n$ is even.

4. Short proof of the Five Color Theorem.
   a) Let $v$ be a 5-vertex in a plane graph $G$. Let $x$ and $y$ be nonadjacent neighbors of $v$, and let $G'$ be the graph obtained from $G$ by contracting the edges $vx$ and $vy$. Prove that if $G'$ is 5-colorable, then $G$ is 5-colorable.
   b) Use part (a) to give a short inductive proof of the Five Color Theorem.

5. Without using the Four Color Theorem, prove that every outerplanar graph is 3-colorable. Apply this to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with $n$ sides, then it is possible to place $n/3$ guards such that every point of the interior is visible to some guard. For $n \geq 3$, construct a polygon that requires $n/3$ guards.

6. The thickness of a graph $G$ is the minimum number of planar graphs needed to partition $E(G)$.
   a) Prove that if $G$ has thickness 2, then $\chi(G) \leq 12$.
   b) For $r$ even and $s$ greater than $(r - 2)^2 / 2$, prove that the thickness of $K_{r,s}$ is $r/2$. 