MATH 412, SPRING 2013 - HOMEWORK 2

WARMUP PROBLEMS: Section 1.2: #3, 6, 8, 10, 12. Section 1.3: #1, 9. Do not write these up! Think about how to solve them to make sure you understand the material before doing the homework.

EXTRA PROBLEMS: Section 1.2: #15, 23, 25, 27, 28, 32, 33, 37, 40, 41. Section 1.3: #10, 14. Do not write these up! These are interesting problems (related to what we have discussed) to provide extra practice.

WRITTEN PROBLEMS: Do five of the six problems below (students registered for four hours or honors must do all six problems). Due Wednesday, January 29.

1. Prove that a graph $G$ is bipartite if and only if every subgraph $H$ of $G$ has an independent set consisting of at least half of $V(H)$.

2. Let $G$ be a simple graph with vertices $v_1, \ldots, v_n$. Let $A^k$ denote the $k$th power of the adjacency matrix of $G$ under matrix multiplication. Prove that entry $i, j$ of $A^k$ is the number of $v_i, v_j$-walks of length $k$ in $G$. Prove that $G$ is bipartite if and only if, for the odd integer $r$ in $\{n - 1, n\}$, the diagonal entries of $A^r$ are all 0. (Reminder: A walk is an ordered list of vertices and edges.)

3. Prove that the graph below cannot be expressed as the union of two cycles.

![Diagram of a graph with seven vertices and six edges]

4. Let $G$ be a graph with girth $g$ in which every vertex has degree at least $k$. Prove that if $k \geq 2$, then $G$ contains a cycle of length at least $(g - 2)(k - 1) + 2$.

5. Use induction on $k$ to prove that every connected simple graph with $2k$ edges decomposes into paths of length 2. Does the conclusion remain true without the hypothesis of connectedness?

6. Let $W$ be a closed walk in a graph $G$. Let $H$ be the subgraph of $G$ consisting of edges used an odd number of times in $W$. Prove that $d_H(v)$ is even for every $v \in V(G)$.