MONOID ACTIONS AND ULTRAFILTER METHODS IN RAMSEY THEORY

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ABSTRACT. First, we prove a theorem on dynamics of actions of monoids by endomorphisms of semigroups. Second, we introduce algebraic structures suitable for formalizing infinitary Ramsey statements and prove a theorem that such statements are implied by the existence of appropriate homomorphisms between the algebraic structures. We make a connection between the two themes above, which allows us to prove some general Ramsey theorems for sequences. We give a new proof of the Furstenberg–Katznelson Ramsey theorem; in fact, we obtain a version of this theorem that is stronger than the original one. We answer in the negative a question of Lupini on possible extensions of Gowers’ Ramsey theorem.

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The main point of the paper is establishing a relationship between monoid actions and Ramsey theory.

In Section 2, we study the dynamics of actions of monoids by continuous endomorphisms on compact right topological semigroups. We outline now the notions relevant to this study and its outcome: (1) a partial order $\mathcal{Y}(M)$; (2) a class of monoids; and (3) the theorem on dynamics of monoid actions.

(1) We associate with each monoid $M$ a partial order $\mathcal{Y}(M)$ on which $M$ acts in an order preserving manner. We define first the order $\mathcal{X}(M)$ consisting of all principle right ideals in $M$, that is, sets of the form $aM$ for $a \in M$, with the order relation $\leq_{\mathcal{X}(M)}$ being inclusion. This order is considered in the representation theory of monoids as in [12]. The monoid $M$ acts on $\mathcal{X}(M)$ by left translations. We then let $\mathcal{Y}(M)$ consist of all non-empty linearly ordered by $\leq_{\mathcal{X}(M)}$ subsets of $\mathcal{X}(M)$. We order $\mathcal{Y}(M)$ by end-extension, that is, we let $x \leq_{\mathcal{Y}(M)} y$ if $x$ is included in $y$ and all elements of $y \setminus x$ are larger with respect to $\leq_{\mathcal{X}(M)}$ than all elements of $x$. The construction of the partial order $\mathcal{Y}(M)$ from the partial order $\mathcal{X}(M)$ is a special case of a set theoretic construction going back to Kurepa [7]. An order preserving action of the monoid $M$ on $\mathcal{Y}(M)$ is induced in the natural way from its action on $\mathcal{X}(M)$.

(2) We introduce a class of monoids we call almost R-trivial, which contains the well known class of R-trivial monoids, see [12], and all the monoids of interest to us. In a monoid $M$, by the **R-class** of $a \in M$ we understand the equivalence class of $a$ with respect to the equivalence relation that makes two elements equivalent if the principle right ideals generated by the two elements coincide, that is, $b_1$ and $b_2$ are equivalent if $b_1M = b_2M$. We call a monoid $M$ **almost R-trivial** if for each element $b$ whose R-class has strictly more than one element we have $ab = b$ for each $a \in M$. (A monoid is R-trivial if the R-class of each element contains only that element.) In Section 2.3, we provide the relevant examples of almost R-trivial monoids.

(3) In Theorem 2.4, which is the main theorem of Section 2, we show that each action of a finite almost R-trivial monoid by continuous endomorphisms on a compact right topological semigroup contains, in a precise sense, the action of $M$ on $\mathcal{Y}(M)$. This result was inspired by Ramsey theoretic considerations, but it may also be of independent dynamical interest.

In Section 3, we introduce new algebraic structures that are appropriate for formalizing various Ramsey statements concerning sequences. We isolate the notions of **basic sequence** and **tame coloring**. In Theorem 3.1, the main theorem of this section, we show that finding a basic sequence on which a given coloring is tame follows from the existence of an appropriate homomorphism. This theorem reduces proving a Ramsey statement to establishing an algebraic property. We introduce a natural notion of tensor product of the algebraic structures studied in this section, which makes it possible to strengthen the conclusion of Theorem 3.1.

In Section 4, we connect the previous two sections with each other and explore Ramsey theoretic issues. In Corollary 4.1, we show that the main result of Section 2...
yields a homomorphism required for the main result of Section 3. This corollary has various Ramsey theoretic consequences. For example, we introduce a notion of *Ramsey monoid* and prove that, among finite almost R-trivial monoids $M$, being Ramsey is equivalent to linearity of the order $\mathcal{X}(M)$. We use this result to show that an extension of Gowers’ Ramsey theorem [4] inquired for by Lupini [8] is false. As other consequences, we obtain some earlier Ramsey results by associating with each of them a finite almost R-trivial monoid. For example, we show the Furstenberg–Katznelson Ramsey theorem for located words, which is stronger than the original version of the theorem from [3]. Our proof is also different from the one in [3].

We state here one Ramsey theoretic result from Section 4, which has Furstenberg–Katznelson’s and Gowers’ theorems, [3], [4], as special instances; see Section 4.3. Let $M$ be a monoid. By a *located word over* $M$ we understand a function from a finite non-empty subset of $\mathbb{N}$ to $M$. For two such words $w_1$ and $w_2$, we write $w_1 \prec w_2$ if the largest element of the domain of $w_1$ is smaller than the smallest element of the domain of $w_2$. In such a case, we write $w_1w_2$ for the located word that is the function whose graph is the union of the graphs of $w_1$ and $w_2$. For a located word $w$ and $a \in M$, we write $a(w)$ for the located word that results from multiplying on the left each value of $w$ by $a$. Given a finite coloring of all located words, we are interested in producing a sequence $w_0 \prec w_1 \prec \cdots$ of located words, for which we control the color of

$$a_0(w_{n_0}) \cdots a_k(w_{n_k}),$$

for arbitrary $a_0, \ldots, a_k \in M$ and $n_0 < \cdots < n_k$. The control over the color is exerted using the partial order $\mathcal{Y}(M)$ introduced above. With each partial order $(P, \leq_P)$, one naturally associates a semigroup $\langle P \rangle$, with its binary operation denoted by $\lor$, that is the semigroup generated freely by the elements of $P$ subject to the relations

$$(1.1) \quad p \lor q = q \lor p = q \text{ if } p \leq_P q.$$  

We consider the semigroup $\langle \mathcal{Y}(M) \rangle$ produced from the partial order $\mathcal{Y}(M)$ in this manner. We now have the following statement, which is proved as Theorem 4.3.

*Let $M$ be almost R-trivial and finite. Fix a finite subset $F$ of the semigroup $\langle \mathcal{Y}(M) \rangle$ and a maximal element $y$ of the partial order $\mathcal{Y}(M)$. For each coloring with finitely many colors of all located words over $M$, there exists a sequence $w_0 \prec w_1 \prec w_2 \prec \cdots$ of located words such that the color of*

$$a_0(w_{n_0}) \cdots a_k(w_{n_k}),$$

*for $a_0, \ldots, a_k \in M$ and $n_0 < \cdots < n_k$, depends only on the element $a_0(y) \lor \cdots \lor a_k(y)$ of $\langle \mathcal{Y}(M) \rangle$ provided that $a_0(y) \lor \cdots \lor a_k(y) \in F$.**
One can view $a_0(y) \lor \cdots \lor a_k(y)$ as the “type” of $a_0(w_{n_0}) \cdots a_k(w_{n_k})$ and the theorem as asserting that the color of $a_0(w_{n_0}) \cdots a_k(w_{n_k})$ depends only on its type. In general, the element $a_0(y) \lor \cdots \lor a_k(y)$ contains much less information than the located word $a_0(w_{n_0}) \cdots a_k(w_{n_k})$, due partly to the disappearance of $w_{n_0}, \ldots, w_{n_k}$ and partly to the influence of relations (1.1).

We comment now on our view of the place of the present work within Ramsey theory. A large portion of Ramsey Theory can be parametrized by a triple $(a, b, c)$, where $a, b, c$ are natural numbers or $\infty$ and $a \leq b \leq c$. (We exclude here, for example, a very important part of Ramsey theory called structural Ramsey theory, for which a general approach is advanced in [6].) The simplest Ramsey theorems are those associated with $a \leq b < \infty = c$. (For example, for each finite coloring of all $a$-element subsets of an infinite set $C$, there exists a $b$-element subset of $C$ such that all of its $a$-element subsets get the same color.) These simplest theorems are strengthened in two directions.

Direction 1: $a \leq b \leq c < \infty$. This is the domain of Finite Ramsey Theory. (For example, for each finite coloring of all $a$-element subsets of a $c$-element set $C$, there exists a $b$-element subset of $C$ such that all of its $a$-element subsets get the same color.) Appropriate structures for this part of the theory are described in [11].

Direction 2: $a = b = c = \infty$. This is the domain of Infinite Dimensional Ramsey Theory. (For example, for each finite Borel coloring of all infinite element subsets of an infinite countable set $C$, there exists an infinite subset of $C$ such that all of its infinite subsets get the same color.) Appropriate structures for this theory were developed in [14] and a General Ramsey Theorem for them was proved there.

The frameworks in 1 and 2 are quite different in particulars, but, roughly speaking, the General Ramsey Theorems (GRT) in both cases have the same form:


Such GRT reduces proving concrete Ramsey statements to proving appropriate pigeonhole principles. In 1, pigeonhole principles are either easy to check directly or, more frequently, they are reformulations of Ramsey statements proved earlier using GRT with the aid of easier pigeonhole principles. So it is a self-propelling system. In 2, pigeonhole principles cannot be obtained this way and they require separate proofs. (The vague reason for this is that the pigeonhole principles here correspond to the case $b = c = \infty$ and $a$ = potential $\infty$.)

This paper can be viewed as providing appropriate structures and general theorems that handle proofs of pigeonhole principles in 2. These structures are quite different from those in 1 and 2.

The concurrently written interesting paper [9] also touches on the theme of ultrafilter methods in Ramsey theory. This work and ours are independent from each other.

2. Monoid actions on semigroups

The theme of this section is, on the face of it, purely dynamical. We study actions of finite monoids on compact right topological semigroups by continuous endomorphisms. We isolate the class of almost $R$-trivial monoids that extends the
well studied class of R-trivial monoids. We prove in Theorem 2.4 that each action of an almost R-trivial finite monoid $M$ on a compact right topological semigroup by continuous endomorphisms contains, in a precise sense, a finite action defined only in terms of $M$. This finite action is an action of $M$ on a partial order $\mathcal{Y}(M)$ introduced in Section 2.1. An important to us reformulation of Theorem 2.4 is done in Corollary 2.7.

2.1. Monoid actions on partial orders. A monoid is a semigroup with a distinguished element that is a left and right identity. By convention, if a monoid acts on a set, the identity element acts as the identity function.

Let $M$ be a monoid. By an $M$-partial order we understand a set $X$ equipped with an action of $M$ and with a partial order $\leq_X$ such that if $x \leq_X y$, then $ax \leq_X ay$, for $x, y \in X$ and $a \in M$. Let $X$ and $Y$ be $M$-partial order. A function $f: X \to Y$ is an epimorphism if $f$ is onto, $f$ is $M$-equivariant, and $\leq_Y$ is the image under $f$ of $\leq_X$. We say that an $M$-partial order $X$ is strong if, for all $y \in X$ and $a \in M$,

$$\{ax \in X : x \leq_X y\} = \{x \in X : x \leq_X ay\}.$$  

For a monoid $M$, consider $M$ acting on itself by multiplication on the left. Set

$$\mathbb{X}(M) = \{aM : a \in M\}$$  

with the order relation being inclusion. Then, $\mathbb{X}(M)$ is an $M$-partial order. We actually have more.

Lemma 2.1. Let $M$ be a monoid. Then $\mathbb{X}(M)$ is a strong $M$-partial order.

Proof. We need to see that if $cM \subseteq abM$, then there is $c'$ such that $c'M \subseteq bM$ and $ac'M = cM$. Since $cM \subseteq abM$, we have $c \in abM$, so $c = abd$ for some $d \in M$. Let $c' = bd$. It is easy to check that this $c'$ works. $\square$

For each finite partial order $X$, let

$$\text{Fr}(X) = \{x \subseteq X : x \neq \emptyset \text{ and } x \text{ is linearly ordered by } \leq_X\}.$$  

The order relation on $\text{Fr}(X)$ is defined by letting for $x, y \in \text{Fr}(X)$,

$$x \leq_{\text{Fr}(X)} y \iff x \subseteq y \text{ and } i <_X j \text{ for all } i \in x \text{ and } j \in y \setminus x.$$  

Observe that $\text{Fr}(X)$ is a forest, that is, the set of predecessors of each each element is linearly ordered. As pointed out by Todorcevic, the operation Fr is a finite version of certain constructions from infinite combinatorics of partial orders [7], [13].

Let $X$ be an $M$-partial order. For $x \in \text{Fr}(X)$ and $a \in M$, let

$$ax = \{ai : i \in x\}.$$  

Clearly, $ax \in \text{Fr}(X)$ and $M \times \text{Fr}(X) \ni (a, x) \to ax \in \text{Fr}(X)$ is an action of $M$ on $\text{Fr}(X)$.

The following lemma is easy to verify.

Lemma 2.2. Let $M$ be a monoid, and let $X$ be an $M$-partial order.

(i) $\text{Fr}(X)$ with the action defined above is a strong $M$-partial order.
(ii) The function $\pi: \text{Fr}(X) \to X$ given by $\pi(x) = \max x$ is an epimorphism between the two $M$-partial orders.

For a finite monoid $M$, set

$$\Upsilon(M) = \text{Fr}(\mathcal{X}(M)).$$

By Lemma 2.2, $\Upsilon(M)$ is a strong $M$-partial order.

2.2. **Compact right topological semigroups.** We recall here some basic notions concerning right topological semigroups.

Let $U$ be a semigroup. As usual, let $E(U)$ be the set of all idempotents of $U$. There is a natural transitive, anti-symmetric relation $\leq_U$ on $U$ defined by

$$u \leq_U v \iff uv = vu = u.$$

This relation is reflexive on the set $E(U)$. So $\leq_U$ is a partial order on $E(U)$.

A semigroup equipped with a topology is called **right topological** if, for each $u \in U$, the function

$$U \ni x \to xu \in U$$

is continuous.

In the proposition below, we collect facts about idempotents in compact semigroups needed here. They are proved in [14, Lemma 2.1, Lemma 2.3 and Corollary 2.4, Lemma 2.11].

**Proposition 2.3.** Let $U$ be a compact right topological semigroup.

(i) $E(U)$ is non-empty.

(ii) For each $v \in E(U)$ there exists a minimal with respect to $\leq_U$ element $u \in E(U)$ with $u \leq_U v$.

(iii) For each minimal with respect to $\leq_U$ element $u \in E(U)$ and each right ideal $I \subseteq X$, there exists $v \in I \cap E(U)$ with $uv = u$.

If $U$ is equipped with a compact topology, that may not interact with multiplication in any way, then there exists the smallest under inclusion compact two-sided ideal of $U$, see [5]. So, for a compact right topological semigroup $U$, let

$$I(U)$$

stand for the smallest compact two-sided ideal with respect to the compact topology on $U$.

2.3. **Almost R-trivial monoids.** Two elements $a, b$ of a monoid $M$ are called **R-equivalent** if $aM = bM$. Of course, by an **R-class** of $a \in M$ we understand the set of all elements of $M$ that are R-equivalent to $a$. A monoid $M$ is called **R-trivial** if each R-class has exactly one element, that is, if for all $a, b \in M$, $aM = bM$ implies $a = b$. This notion with an equivalent definition was introduced in [10]. For the role of R-trivial monoids in the representation theory of monoids see [12, Chapter 2].
Note that if $M$ is R-trivial, then the partial order $\mathcal{X}(M)$ can be identified with $M$ taken with the partial order $a \leq_M b$ if and only if $a \in bM$. We call a monoid $M$ almost R-trivial if, for each $b \in M$ whose R-class has more than one element, we have $ab = b$ for all $a \in M$.

We present now examples of almost R-trivial monoids relevant in Ramsey theory.

**Examples.** 1. Let $n \in \mathbb{N}, n > 0$. Let

$$G_n = \{0, \ldots, n-1\}$$

be multiplication defined by

$$i \cdot j = \min(i + j, n - 1).$$

We set $1_{G_n} = 0$.

The monoid $G_n$ is R-trivial since, for each $i \in G_n$, we have $iG_n = \{i, \ldots, n-1\}$.

The monoid $G_n$ is associated with Gowers' Ramsey theorem [4], see also [14].

2. Fix $n \in \mathbb{N}, n > 0$. Let

$$I_n$$

be the set of all non-decreasing functions that map $n$ onto some $k \leq n$. These are precisely the non-decreasing functions $f : n \to n$ such that $f(0) = 0$ and $f(i+1) \leq f(i) + 1$ for all $i < n - 1$. The multiplication operation is composition and 1 is the identity function from $n$ to $n$.

The monoid $I_n$ is R-trivial. To see this, let $f, g \in I_n$ be such that $f \in gI_n$ and $g \in fI_n$, that is, $f = g \circ h_1$ and $g = f \circ h_2$, for some $h_1, h_2 \in I_n$. It follows from these equations that $f(i) \leq g(i)$, for all $1 \leq i \leq n$, and $g(i) \leq f(i)$, for all $1 \leq i \leq n$. Thus, $f = g$.

The monoid $I_n$ is associated with Lupini's Ramsey theorem [8].

3. Fix two disjoint sets $A, B$, and let 1 not be an element of $A \cup B$. Let

$$J(A, B)$$

be $\{1\} \cup A \cup B$. Define multiplication on $J(A, B)$ by letting, for each $c \in A \cup B$, $c \cdot a = c$, if $a \in A$;

$$c \cdot b = b$$

if $b \in B$.

Of course, we define $1 \cdot c = c \cdot 1 = c$ for all $c \in J(A, B)$. We leave it to the reader to check that so defined multiplication is associative.

The monoid $J(A, B)$ is almost R-trivial. Indeed, a quick check gives, for $a \in A$ and $b \in B$,

$$aJ(A, B) = \{a\} \cup B, bJ(A, B) = B, 1J(A, B) = J(A, B).$$

Thus, the only elements of $J(A, B)$, whose R-classes can possible have size bigger than one, are elements of $B$. But for all $c \in J(A, B)$ and $b \in B$, we have $cb = b$. It follows that $J(A, B)$ is almost R-trivial (and not R-trivial if the cardinality of $B$ is strictly bigger than one).

The monoid $J(\emptyset, B)$ for a one element set $B$ is associated with Hindman’s theorem, see [14], and for arbitrary finite $B$ with the infinitary Hales–Jewett theorem,
see [14]. For arbitrary finite $A$ and $B$, $J(A,B)$ is associated with the Furstenberg–
Katznelson theorem [3].

2.4. The theorem on monoid actions. In the results of this section, we obey
the following conventions:

— $U$ is a compact right topological semigroup;
— $M$ is a finite monoid acting on $U$ by continuous endomorphisms.

The following theorem is the main result of this section.

Theorem 2.4. Assume $M$ is almost $R$-trivial. There exists a function $g: \mathcal{Y}(M) \to E(U)$ such that

(i) $g$ is $M$-equivariant;
(ii) $g$ is order reversing with respect to $\leq_{\mathcal{Y}(M)}$ and $\leq_U$;
(iii) $g$ maps maximal elements of $\mathcal{Y}(M)$ to $I(U)$.

Moreover, if $\mathcal{Y}(M)$ has at most two elements, then $g$ maps maximal elements of
$\mathcal{Y}(M)$ to minimal elements of $U$.

We will need the following lemma.

Lemma 2.5. Let $F$ be a strong $M$-partial order that is a forest. Assume that
$f: F \to U$ is $M$-equivariant. Then there exists $g: F \to E(U)$ such that

(i) $g$ is $M$-equivariant;
(ii) $g$ is order reversing with respect to $\leq_F$ and $\leq_U$;
(iii) $g^{-1}(I(U))$ contains $f^{-1}(I(U))$.

Proof. Let $A \subseteq F$ be downward closed. Assume there exists $g_A: F \to E(U)$ such that
(i) and (iii) hold and additionally, for all $i,j \in A$,

(*) if $i <_F j$, then $g_A(j)g_A(i) = g_A(j)$.

Note the condition that the values of $g_A$ are in $E(U)$, so they are idempotents. Let
$B \subseteq F$ be such that $A \subseteq B$ and all the immediate predecessors of elements of $B$
are in $A$. We claim that there exists $g_B: F \to E(U)$ fulfilling (i), (iii), (*) for all
$i,j \in B$, and $g_B \upharpoonright A = g_A \upharpoonright A$.

First, define $g_B': F \to U$ by letting, for $j \in F$,

$$g_B'(j) = g_A(i_k)g_A(i_k-1)\cdots g_A(i_1),$$

where $i_1 <_F \cdots <_F i_k$ lists the set $\{i \in F: i \leq_F j\}$ is the increasing order.

We check that $g_B'$ fulfills (i), (iii), and (*) for $i,j \in B$. Point (i) holds since for
each $a \in M$ we have

$$a(g_B'(j)) = a(g_A(i_k))a(g_A(i_{k-1}))\cdots a(g_A(i_1))$$

$$= g_A(a(i_k))g_A(a(i_{k-1}))\cdots g_A(a(i_1)) = g_B'(a(j)),$$

with the second equality holding since the function $g_A$ is $M$-equivariant and the
third one holding by idempotence of the values of $g_A$ and the fact that $F$ is and
$M$-tree. Point (iii) holds since $I(U)$ is a right ideal and the function $g_A$ fulfills (iii).
To check (*) for $i,j \in B$ with $i <_F j$, let

$$i_1 <_F \cdots <_F i_k <_F \cdots <_F i_l$$

list all the predecessors of $j$ in the increasing order so that $i_k = i$ and, of course, $i_i = j$. Then, since $j \in B$, we have $i_1, \ldots, i_k \in A$ and, therefore, we get

$$g_B'(j) = g_A(i_1) \cdots g_A(i_k) = g_A(j) \cdots g_A(i).$$

By the same computation carried out for $i = j \in A$, we see

$$g_B'(i) = g_A(i), \quad \text{for } i \in A.$$

It follows that

$$g_B'(j)g_B'(i) = g_A(j) \cdots g_A(i)g_A(i) = g_A(j) \cdots g_A(i) = g_B'(j).$$

This equality shows that $(\ast)$ holds for $i, j \in B$. Finally, note that (2.4) implies that $g_B' | A = g_A | A$. Thus, $g_B'$ has all the desired properties.

To construct $g_B$ from $g_B'$, consider $U_F$ with coordinateswise multiplication and the product topology. This is a right topological semigroup. Define $H \subseteq U_F$ to consist of all $x \in U_F$ such that

$(\alpha)$ the function $F \ni i \to x_i \in U$ fulfills (i), (iii), and $(\ast)$ for $i, j \in B$ and

$(\beta)$ $x_i = g_A(i)$ for all $i \in A$.

First we observe that $H$ is a subsemigroup of $U_F$. Condition (i) is clearly closed under multiplication. Condition (iii) is closed under multiplication since $I(U)$ is a two-sided ideal. Condition (\ast) is closed under multiplication in the presence of $(\beta)$ since, for $x, y \in H$ and $i, j \in B$ with $i <_F j$, we have $i \in A$ and, therefore,

$$x_jy_jx_iy_i = x_jy_jg_A(i)g_A(i) = x_jy_jy_iy_i = x_jy_j.$$

This verification shows that $(\alpha)$ is closed under multiplication in the presence of $(\beta)$. Condition $(\beta)$ is closed under multiplication since $g_A(i)$ is an idempotent.

Next note that $H$ is compact since all conditions defining $H$ are clearly topologically closed with a possible exception of $(\ast)$ for $i, j \in B$ with $i <_F j$. Note that in this case $i \in A$. Since $x \in U_F$ and $i \in A$, we have $x_i = g_A(i)$, condition $(\ast)$ translates to $x_jg_A(i) = x_j$ for $i \in A$ and $j \in B$ with $i <_F j$. This condition is closed since $U$ is right topological. Finally note that $H$ is non-empty since $g_B'$ is its element. By Ellis’ theorem $H$ contains an idempotent. Let $g_B \in H$ be such an idempotent. It has all the required properties.

The above procedure describes the passage from $A$ to $B$ if all immediate predecessors of elements of $B$ are in $A$. We now define $g_B$. Note first that $f$ fulfills (i), (iii), and $(\ast)$ for $A = \emptyset$, with the last condition holding vacuously.

We apply the above claim recursively starting with $A = \emptyset$. After performing this procedure, we end up with a function $g_F: F \to U$ such that (i), (iii) hold as does $(\ast)$ for all $i, j \in F$. So $f$ has all the required properties except its values may not be in $E(U)$. To remedy this shortcoming, consider again the compact right topological semigroup $U_F$ with coordinateswise multiplication and the product topology. Define $H \subseteq U_F$ to consist of all $x \in U_F$ such that the function $F \ni i \to x_i \in U$ fulfills (i) and (iii) and, vacuously, $(\ast)$ for $i, j \in \emptyset$. Then $H$ is non-empty since $f \in H$. Let $g_0$ be an idempotent on $H$. Then $g_0$ is as required.

Starting with $g_0$ and recursively using the above procedure of going from $g_A$ to $g_B$, we produce $g_F: F \to E(U)$ fulfilling (i), (iii) and $(\ast)$ for all $i, j \in F$. Note that
since the values of $g_{F}$ are idempotents, condition (⋆) implies (ii), which finishes the proof of the lemma.

Lemma 2.6. Assume that $ab = b$, for all $a, b \in M$ with $b \neq 1_{M}$. Then there exists minimal $u_{1} \in E(U)$ such that

(i) $u_{1} \in I(U)$
(ii) $a(u_{1}) = b(u_{1})$, for all $a, b \in M$ with $a \neq 1_{M} \neq b$.

Proof. Observe that, for $a, b \in M \setminus \{1_{M}\}$, since $ba = a$, we have

$$a(U) = ba(U) = b(a(U)) \subseteq b(U).$$

By symmetry, we see that $a(U) = b(U)$. Let $T$ be the common value of the images of $U$ under the elements of $M \setminus \{1_{M}\}$. Clearly $T$ is a compact subsemigroup of $U$.

Note that

$$a(u) = u, \text{ for } u \in T, a \in M.$$ 

Let $u_{0} \in T$

be a minimal with respect to $\leq T$ idempotent.

Let $u_{1} \in U$ be a minimal idempotent in $U$ with $u_{1} \leq U u_{0}$. Since $u_{1}$ is minimal, we have

$$u_{1} \in I(U).$$

We show that

$$a(u_{1}) = u_{0}, \text{ for all } a \in M \setminus \{1_{M}\}.$$ 

Indeed, since $u_{1} \leq U u_{0}$ and $u_{0} \in T$, by (2.5), we get

$$a(u_{1}) \leq U a(u_{0}) = u_{0}.$$ 

Thus, $a(u_{1}) \leq T u_{0}$ and $a(u_{1}) \in T$. Since $u_{0}$ is minimal in $T$, we get $a(u_{1}) = u_{0}$.

Equations (2.6) and (2.7) show that $u_{1}$ is as required.

Proof of Theorem 2.4. Let

$$B = \{b \in M : ab = b \text{ for all } a \in M\}.$$ 

Note that $M' = \{1_{M}\} \cup B$ is a monoid fulfilling the assumption of Lemma 2.6. Let $u_{1} \in U$ be an element as in the conclusion of Lemma 2.6.

Define a function $h : M \rightarrow U$ by $h(a) = a(u_{1})$. Note that $h$ is $M$-equivariant if $M$ is taken with left multiplication action. Observe the following two implications:

(a) if $a_{1} \in M \setminus B$, $a_{2} \in M$, and $a_{1}M = a_{2}M$, then $a_{1} = a_{2}$;
(b) if $a \in M$ and $b \in B$, then $bM \subseteq aM$.

Point (a) follows from $M$ being almost R-trivial. Point (b) is a consequence of $b = ab \in aM$. Let $\rho : M \rightarrow \mathcal{X}(M)$ be the equivariant surjection $\rho(a) = aM$. Note that by (a) and (b), $\rho$ is injective on $M \setminus B$, all points in $B$ are mapped to a single point of $\mathcal{X}(M)$ that is the smallest point of this partial order, and no point of $M \setminus B$ is mapped to this smallest point. It now follows from the properties of $u_{1}$ listed in Lemma 2.6 that $h$ factors through $\rho$ giving a function $h' : \mathcal{X}(M) \rightarrow U$ with
h' \circ \rho = h. Since \rho and h are M-equivariant, so is h'. Let \pi: \mathcal{Y}(M) \to \mathcal{X}(M) be the M-equivariant function given by Lemma 2.2(ii). Then f: \mathcal{Y}(M) \to U, defined by f = h' \circ \pi, is M-equivariant. Furthermore, since u_1 \in I(U) gives h(1_M) \in I(U), and hence h'(1_M M) \in I(U), we see that the maximal elements of \mathcal{Y}(M) are mapped by f to I(U). Note that if \mathcal{X}(M) has at most two elements, then we can let g = f. Then h'(1_M M) is a minimal with respect to \leq_U idempotent, and g maps all maximal elements of \mathcal{Y}(M) to h'(1_M M). Without any restrictions on the size of \mathcal{X}(M), Lemma 2.5 can be applied to f giving a function g as required by points (i)–(iii).

2.5. Semigroups from partial orders and a restatement of the theorem.

For a partial order P, let

\langle P \rangle

be the semigroup, whose binary operation is denoted by \lor, generated freely by elements of P modulo the relations

\begin{equation}
    p \lor q = q \lor p = q, \text{ for } p, q \in P \text{ with } p \leq_P q.
\end{equation}

(2.8)

That is, each element of \langle P \rangle can be uniquely written as \lor_{i=0}^{n} p_i for some \( n \in \mathbb{N} \) and with \( p_i \) and \( p_{i+1} \) being incomparable with respect to \( \leq_P \), for all \( 0 < i < n \). Note that if P is linear, then \langle P \rangle = P.

Observe that if M is a monoid and P is an M-partial order, then the action of M on P naturally induces an action of M on \langle P \rangle by endomorphisms.

A moment of thought convinces one that the function from Theorem 2.4 extends to a homomorphism from \langle \mathcal{Y}(M) \rangle to U—condition (ii) of Theorem 2.4 and the fact that the function in that theorem takes values in \mathcal{E}(U) are responsible for this. Therefore, we get the following corollary, which we state with the conventions of Section 2.4.

**Corollary 2.7.** Assume M is almost R-trivial. There exists an M-equivariant homomorphism of semigroups \( g: \langle \mathcal{Y}(M) \rangle \to U \) that maps maximal elements of \( \mathcal{Y}(M) \) to \( I(U) \).

3. Infinite Ramsey theorems

The goal of this section is purely Ramsey theoretic. We introduce structures, we call \Lambda-partial semigroups, that generalize the partial semigroup setting of [2]. For such structures, we introduce the notion of basic sequence. Basic sequences appear in Ramsey statements whose aim it is to control coloring on them. We introduce a new general notion of controlling a coloring on a basic sequence. The main result then is Theorem 3.1, which gives such a control over a coloring on a basic sequence from the existence of an appropriate homomorphism. Thus, proving Ramsey statements is reduced to finding homomorphisms. Furthermore, we introduce a natural notion of tensor product for \Lambda-semigroups that allows us to propagate the existence of homomorphisms and, therefore, to propagate Ramsey statements.
3.1. Λ-partial semigroups and Λ-semigroups. Here we recall the notion of
partial semigroup and, more importantly, we introduce our main Ramsey theoretic
structures: Λ-partial semigroups, and Λ-semigroups and homomorphisms between
them.

A partial semigroup is a set $S$ with a function (operation) from a subset of
$S \times S$ to $S$ such that for all $r, s, t \in S$ if one of the two products $(rs)t$, $r(st)$ is defined,
then so is the other and $(rs)t = r(st)$. (This condition is slightly stronger than what
is assumed in [2] and [14]: if both products $(rs)t$ and $r(st)$ are defined, then they
are equal. We motivate our choice as follows: on the one hand, no examples are
lost by assuming the stronger condition, on the other hand, in calculations, we are
spared the task of keeping track of the distribution of parentheses. Note, however,
that everything that follows can be done using only the weaker condition.) Note
that a semigroup is a partial semigroup whose binary operation is total.

Now, let $Λ$ be a set. Let $S$ be a partial semigroup and let $X$ be a set. By a
Λ-partial semigroup over $S$ based on $X$ we understand an assignment to each
$λ \in Λ$ of a partial function, which we also call $λ$, from a subset of
$X$ to $S$ such that for all $s_0, \ldots, s_k \in S$ there exists $x \in X$ such that, for each $λ \in Λ$, $λ(x)$ is defined
and $s_0λ(x), \ldots, s_kλ(x)$ are all defined. We call it a Λ-semigroup over
$S$ based on $X$ if $S$ is a semigroup and the domain each $λ \in Λ$ is equal to $X$, that is, $λ$ is
a total function. We call a Λ-partial semigroup point based if $X$ consist of one
point, which we usually denote by $•$; so $X = \{•\}$ in this case.

We give now some constructions that will be used in Section 4. Let $S$ be a partial
semigroup. A function $h: S \to S$ is an endomorphism if for all $s_1, s_2 \in S$ with
$s_1s_2$ defined, $h(s_1)h(s_2)$ is defined and $h(s_1s_2) = h(s_1)h(s_2)$. Let $M$ be a monoid.
An action of $M$ on $S$ is called is called an action of $M$ on $S$ by endomorphisms
if, for each $a \in M$, the function $s \to a(s)$ is an endomorphism of $S$ and, for all
$s_1, \ldots, s_n \in S$ and each $a \in M$, there is $t \in S$ such that $s_1a(t), \ldots, s_na(t)$ is defined.
Obviously, we will identify such an action with the function $α: M \times S \to S$ given
by $α(a, s) = a(s)$. Let

\[(3.1)\]

be the $M$-partial semigroup over $S$ obtained by interpreting each $a \in M$
as the function from $S$ to $S$ given by the action, that is,

$S \ni s \to α(a, s) \in S$.

Fix now $s_0 \in S$. Let

\[(3.2)\]

be the point based $M$-partial semigroup over $S$ obtained by interpreting each $a \in M$
as the function on $\{•\}$ given by

$a(•) = α(a, s_0)$.

The Λ-partial semigroups used in this paper will be of the above form or will be
obtained from such by the tensor product operation defined in Section 3.5.
3.2. **Basic sequences and tame colorings.** Assume we have a Λ-partial semigroup over $S$ and based on $X$. A sequence $(x_n)$ of elements of $X$ is called **basic** if for all $n_0 < \cdots < n_l$ and $\lambda_0, \ldots, \lambda_l \in \Lambda$ the product

$$\lambda_0(x_{n_0}) \lambda_1(x_{n_1}) \lambda_2(x_{n_2}) \cdots \lambda_l(x_{n_l})$$

is defined in $S$.

Assume we have a point based Λ-semigroup $A$ over $S$. To make notation clearer, we use $\lor$ for the binary operation on $A$. We say that a coloring of $S$ is **A-tame on** $(x_n)$, where $(x_n)$ is a basic sequence, if the color of the elements of the form (3.3) with the additional condition

$$\lambda_k(\bullet) \lor \cdots \lor \lambda_l(\bullet) \in \Lambda(\bullet), \text{ for each } k \leq l,$$

depends only on the element $\lambda_0(\bullet) \lor \cdots \lor \lambda_l(\bullet)$ of $A$.

3.3. **Λ-semigroups from Λ-partial semigroups.** There is a canonical way of associating a Λ-semigroup to each Λ-partial semigroup. Let $\gamma X$ be the set of all ultrafilters $U$ on $X$ such that for each $s \in S$ and $\lambda \in \Lambda$

$$\{x \in X : s\lambda(x) \text{ is defined}\} \in U.$$ 

It is clear that $\gamma X$ is compact and non-empty. Each $\lambda$ extends to a function, again called $\lambda$, from $\gamma X$ to $\beta S$ by the usual formula, for $U \in \gamma X$,

$$A \in \lambda(U) \iff \lambda^{-1}(A) \in U.$$ 

It is easy to see that the image of each $\lambda$ is included in $\gamma S$. Since $\gamma S$ is a semigroup, we get a Λ-semigroup.

3.4. **The Ramsey theorem.** The following notion will be crucial in stating Theorem 3.1. Assume we have Λ-semigroups, $A$ and $B$, with $A$ being over $A$ and based on $X$ and $B$ being over $B$ and based on $Y$. A **homomorphism from $A$ to $B$** is a pair of functions $f, g$ such that $f : X \to Y$, $g : A \to B$, $g$ is a homomorphism of semigroups, and, for each $x \in X$ and $\lambda \in \Lambda$, we have

$$\lambda(f(x)) = g(\lambda(x)).$$

The following theorem is the main result of Section 3.

**Theorem 3.1.** Fix a finite set $\Lambda$. Let $S$ be a Λ-partial semigroup over $S$, and let $A$ be a point based Λ-semigroup. Let $(f, g) : A \to \gamma S$ be a homomorphism. Then for each $D \in f(\bullet)$ and each finite coloring of $S$, there exists a basic sequences $(x_n)$ of elements of $D$ on which the coloring is $A$-tame.

**Proof.** Let $S$ be based on a set $X$. Set $\mathcal{U} = f(\bullet)$. Observe that if $\lambda(\bullet) = \lambda'(\bullet)$, then $\lambda(\mathcal{U}) = \lambda'(\mathcal{U})$ since

$$\lambda(f(\bullet)) = g(\lambda(\bullet)) = g(\lambda'(\bullet)) = \lambda'(f(\bullet)).$$

This observation allows us to define for $\sigma \in \Lambda(\bullet)$,

$$\sigma(\mathcal{U}) = \lambda(\mathcal{U})$$
Indeed, fix $\lambda \in \Lambda$ with $\sigma = \lambda(\bullet)$. Then we have

$$g(\sigma) = \sigma(U).$$

(3.5)

For $P \subseteq X$ and $\sigma \in \Lambda(\bullet)$, set

$$\sigma(P) = \bigcap \{ \lambda(P) : \lambda(\bullet) = \sigma \}.$$

Note that if $P \in U$ and $\lambda \in \Lambda$, then $\lambda(P) \in \lambda(U)$ since $P \subseteq \lambda^{-1}(\lambda(P))$. So, for $\lambda$ with $\lambda(\bullet) = \sigma$, we have $\lambda(P) \in \sigma(U)$, and, therefore, by finiteness of $\Lambda$, we get $\sigma(P) \in \sigma(U)$.

Consider a finite coloring of $S$. Let $P \in U$ be such that the coloring is constant on $\sigma(P)$ for each $\sigma \in \Lambda(\bullet)$, using the obvious observation that $\sigma(P) \subseteq \sigma(P')$ if $P \subseteq P'$.

Now, we produce $x_n \in X$ and $P_n \subseteq X$ so that

(i) $x_n \in D$, $P_n \subseteq P$;

(ii) $\lambda_1(x_m)\lambda_2(u) \in (\lambda_1(\bullet) \lor \lambda_2(\bullet))(P_m)$, for all $m_1 < n$, all $u \in P_n$, and all $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1(\bullet) \lor \lambda_2(\bullet) \in \Lambda(\bullet)$;

(iii) $\forall u \lambda_1(x_m)\lambda_2(u) \in (\lambda_1(\bullet) \lor \lambda_2(\bullet))(P_m)$, for all $m \leq n$ and all $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1(\bullet) \lor \lambda_2(\bullet) \in \Lambda(\bullet)$;

(iv) $\lambda(x_m) \in \lambda(\bullet)(P_m)$, for all $m \leq n$ and all $\lambda \in \Lambda$.

Note that in points (ii) and (iii) above the condition $\lambda_1(\bullet) \lor \lambda_2(\bullet) \in \Lambda(\bullet)$ ensures that $(\lambda_1(\bullet) \lor \lambda_2(\bullet))(P_m)$ and $(\lambda_1(\bullet) \lor \lambda_2(\bullet))(P_m)$ are defined.

Assume we have $x_m, P_m$ for $m < n$ as above. We produce $x_n$ and $P_n$ so that points (i)–(iv) above hold. Define $P_n$ by letting

$$P_n = P \cap \bigcap_{m < n} C_m,$$

where $C_m$ consists of those $u \in X$ for which

$$\forall \lambda_1, \lambda_2 \in \Lambda \ (if \ \lambda_1(\bullet) \lor \lambda_2(\bullet), \ then \ \lambda_1(x_m)\lambda_2(u) \in (\lambda_1(\bullet) \lor \lambda_2(\bullet))(P_m)).$$

For $n = 0$, by convention, we set $\bigcap_{m < n} C_m = S$. Observe that (ii) holds for $n$. Our inductive assumption (iii) implies that $C_m \in U$. Thus, the definition of $P_n$ gives that $P_0 = P \in U$ and, for $n > 0$, $P_n \in U$.

Using (3.5) in the last equality, we have that, for all $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1(\bullet) \lor \lambda_2(\bullet) \in \Lambda(\bullet)$,

$$\lambda_1(U) \ast \lambda_2(U) = \lambda_1(f(\bullet)) \ast \lambda_2(f(\bullet)) = g(\lambda_1(\bullet)) \ast g(\lambda_2(\bullet))$$

$$= g(\lambda_1(\bullet) \lor \lambda_2(\bullet)) = (\lambda_1(\bullet) \lor \lambda_2(\bullet))(U).$$

(3.6)

Separately, we note that $P_n \in U$ and therefore, for $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1(\bullet) \lor \lambda_2(\bullet) \in \Lambda(\bullet)$,

$$\lambda_1(\bullet)(P_n) \in \bigcap_{m < n} \lambda_1(\bullet)(P_n) \subseteq \Lambda(\bullet), \ and \ \lambda_1(\bullet)(P_n) \in \lambda_1(\bullet)(U) = \lambda_1(U).$$

(3.7)
It follows from (3.6) and (3.7) that we can pick \( x_n \) for which (iii) and (iv) hold. Since \( D \in \mathcal{U} \), we can also arrange that \( x_n \in D \). So (i) is also taken care of.

Now, it suffices to show that the sequence \( (x_n) \) constructed above is as needed. The entries of \( (x_n) \) come from \( D \) by (i). By induction on \( l \), we show that for all \( m_0 < m_1 < \cdots < m_l \) and all \( \lambda_0, \lambda_1, \ldots, \lambda_l \in \Lambda \), we have

\[
\lambda_0(x_{m_0})\lambda_1(x_{m_1})\lambda_2(x_{m_2})\cdots\lambda_l(x_{m_l}) \in (\lambda_0(\bullet) \lor \lambda_1(\bullet) \lor \cdots \lor \lambda_l(\bullet))(P_{m_0}),
\]

provided that \( \lambda_k(\bullet) \lor \cdots \lor \lambda_l(\bullet) \in \Lambda(\bullet) \) for all \( k \leq l \). This claim will establish the theorem since \( P_{m_0} \subseteq P \) by (i).

The case \( l = 0 \) of (3.8) is (iv). We check the inductive step for (3.8) using point (ii). Let \( l > 0 \). Fix \( m_0 < m_1 < \cdots < m_l \) and \( \lambda_0, \lambda_1, \ldots, \lambda_l \in \Lambda \). By our inductive assumption, we have

\[
\lambda_1(x_{m_1})\lambda_2(x_{m_2})\cdots\lambda_l(x_{m_l}) \in (\lambda_1(\bullet) \lor \cdots \lor \lambda_l(\bullet))(P_{m_1}).
\]

Let \( \lambda \in \Lambda \) be such that

\[
\lambda(\bullet) = \lambda_1(\bullet) \lor \cdots \lor \lambda_l(\bullet).
\]

Since

\[
(\lambda_1(\bullet) \lor \cdots \lor \lambda_l(\bullet))(P_{m_1}) \subseteq \lambda(P_{m_1}),
\]

by (3.9), there exists \( y \in P_{m_1} \) such that

\[
\lambda(y) = \lambda_1(x_{m_1})\lambda_2(x_{m_2})\cdots\lambda_l(x_{m_l}).
\]

Since \( m_0 < m_1 \) and since \( y \in P_{m_1}, \) from (ii) with \( n = m_1 \), we get

\[
\lambda_0(x_{m_0})\lambda(y) \in (\lambda_0(\bullet) \lor \lambda(\bullet))(P_{m_0}).
\]

Note that (ii) can be applied here as \( \lambda_0(\bullet) \lor \lambda(\bullet) \in \Lambda(\bullet) \) as

\[
\lambda_0(\bullet) \lor \lambda(\bullet) = \lambda_0(\bullet) \lor \lambda_1(\bullet) \lor \cdots \lor \lambda_l(\bullet).
\]

Now (3.8) follows from (3.12) together with (3.10) and (3.11). \( \Box \)

In the proof above, at stage \( n \), \( x_n \) is chosen arbitrarily from sets belonging to \( f(\bullet) \). It follows that if \( f(\bullet) \) is assumed to be non-principal, then the sequence \( (x_n) \) can be chosen to be injective.

### 3.5. Tensor product of \( \Lambda \)-partial semigroups

We introduce and apply a natural notion of tensor product for \( \Lambda \)-semigroups.

Let \( \Lambda_0, \Lambda_1 \) be sets. Let

\[
\Lambda_0 \ast \Lambda_1 = \Lambda_0 \cup \Lambda_1 \cup (\Lambda_0 \times \Lambda_1),
\]

where the union is taken to be disjoint. Fix a good partial semigroup \( S \). Let \( S_i, i \leq 1, \) be \( \Lambda_i \)-partial semigroups over \( S \). Let \( S_i \) be based on \( X_i \). Define

\[
S_0 \otimes S_1
\]

to be the \( \Lambda_0 \ast \Lambda_1 \)-partial semigroup over \( S \) based on \( X_0 \times X_1 \) such that with \( \lambda_0 \in \Lambda_0, \lambda_1 \in \Lambda_1, \) and \( (\lambda_0, \lambda_1) \in \Lambda_0 \times \Lambda_1 \) we associate functions from \( X_0 \times X_1 \) to \( S \) by letting

\[
\lambda_0(x_0, x_1) = \lambda_0(x_0), \quad \lambda_1(x_0, x_1) = \lambda_1(x_1), \quad (\lambda_0, \lambda_1)(x_0, x_1) = \lambda_0(x_0)\lambda_1(x_1),
\]
where the product on the right hand side is computed in $S$ and the left hand side is defined whenever the product is. It is easy to check that for each $s_0, \ldots, s_m \in S$ and each $\vec{\lambda} \in \Lambda_0 \otimes \Lambda_1$ there exists $\vec{x} \in X_0 \times X_1$ such that $s_j \vec{\lambda}(\vec{x})$ is defined for each $j \leq m$, so $S_0 \otimes S_1$ is indeed a $\Lambda_0 \otimes \Lambda_1$-partial semigroup.

It is clear that if each $A_i$, $i < n$, is a $\Lambda_i$-semigroup, then $\otimes_{i<n} A_i$ is a $\Lambda_{<n}$-semigroup. Note that if each $A_i$ is point based, then so is the tensor product.

**Proposition 3.2.** Let $S$ be a partial semigroup. Let $S_i$, $i = 0, 1$, be $\Lambda_i$-partial semigroups over $S$. Then there is a homomorphism

$$\gamma S_0 \otimes \gamma S_1 \to \gamma(S_0 \otimes S_1).$$

**Proof.** Let $S_0$ be based on $X_0$ and $S_1$ on $X_1$. Then $\gamma S_0 \otimes \gamma S_1$ is based on $\gamma X_0 \times \gamma X_1$, while $\gamma(S_0 \otimes S_1)$ on $\gamma(X_0 \times X_1)$. Consider the natural map $\gamma X_0 \times \gamma X_1 \to \gamma(X_0 \times X_1)$ given by

$$(U, V) \to U \times V,$$

where, for $C \subseteq X_0 \times X_1$,

$$C \in U \times V \iff \{x_0 \in X_0 : \{x_1 \in X_1 : (x_0, x_1) \in C\} \in V\} \in U.$$

Then

$$(f, \text{id}_S) : \gamma S_0 \otimes \gamma S_1 \to \gamma(S_0 \otimes S_1),$$

where $f(U, V) = U \times V$, is the desired homomorphism. \qed

**Proposition 3.3.** Fix semigroups $A$ and $B$. For $i = 0, 1$, let $A_i$ and $B_i$ be $\Lambda_i$-semigroups over $A$ and $B$, respectively. Let $(f_i, g_i) : A_i \to B_i$ be homomorphisms. Then

$$(f_0 \times f_1, g_0 \times g_1) : A_0 \otimes A_1 \to B_0 \otimes B_1$$

is a homomorphism.

**Proof.** Let $A_i$ be based on a set $X_i$. For $\vec{x} \in \prod_{i<n} X_i$ and $\vec{\lambda} \in \prod_{i<n} \Lambda_i$, we have

$$g(\vec{\lambda}(\vec{x})) = g(\lambda_0(x_0) \cdots \lambda_{n-1}(x_{n-1})) = g(\lambda_0(x_0)) \cdots g(\lambda_{n-1}(x_{n-1}))$$

$$= \lambda_0(f_0(x_0)) \cdots \lambda_{n-1}(f_{n-1}(x_{n-1})) = \vec{\lambda}((\prod_{i<n} f_i)(\vec{x})).$$

where the second equality holds since $g$ is a homomorphism of semigroups and the third equality holds since each $(f_i, g_i)$ is a homomorphism from $A_i$ to $B_i$. This check finishes the proof. \qed

3.6. **Propagation of homomorphisms.** The first application has to do with relaxing condition (3.4). This is done in condition (3.13).

Let $A$ be a point based $\Lambda$-semigroup based on a semigroup $A$. As before, we denote by $\lor$ the binary operation on $A$. Let $F$ be a subset of $A$, let $S$ be a $\Lambda$-partial semigroup based on a partial semigroup $S$, and let $(x_n)$ be a basic sequence in $S$. A coloring of $S$ is said to be $F$-$A$-tame on $(x_n)$ if the color of elements of the form (3.3) with the additional condition

$$(3.13) \quad \lambda_k(\bullet) \lor \cdots \lor \lambda_l(\bullet) \in F, \text{ for each } k \leq l,$$

depends only on the element $\lambda_0(\bullet) \lor \cdots \lor \lambda_l(\bullet)$ of $A$. 

The following corollary is a strengthening of Theorem 3.1, but it follows from that theorem via the tensor product construction.

**Corollary 3.4.** Fix a finite set \( \Lambda \) and a finite subset \( F \) of a semigroup \( A \). Let \( S \) be a \( \Lambda \)-partial semigroup, let \( A \) be a point based \( \Lambda \)-semigroup over \( A \), and let \((f,g) : A \to \gamma S\) be a homomorphism. Then for each \( D \in f(\bullet) \) and each finite coloring of \( S \), there exists a basic sequences \((x_n)\) of elements of \( D \) on which the coloring is \( F \)-\( A \)-tame.

**Proof.** Fix a natural number \( r > 0 \). By

\[
\Lambda_{<r} \text{ and } \Lambda_{<\infty}
\]

we denote the set of all sequences \( \vec{\lambda} = (\lambda_0, \ldots, \lambda_m) \) of elements of \( \Lambda \) with \( m < r \), and and with an arbitrary \( M \), respectively. We associate with each such \( \vec{\lambda} \) an element \( \vec{\lambda}(\bullet) \) of \( A \) by letting

\[
\vec{\lambda}(\bullet) = \lambda_0(\bullet) \lor \cdots \lor \lambda_m(\bullet).
\]

Since \( F \) is finite, there exists \( r \) such that

\[
F \cap \{ \vec{\lambda}(\bullet) : \vec{\lambda} \in \Lambda_{<\infty} \} \subseteq \{ \vec{\lambda}(\bullet) : \vec{\lambda} \in \Lambda_{<r} \}.
\]

Thus, it suffices to show the corollary for \( F = \{ \vec{\lambda}(\bullet) : \vec{\lambda} \in \Lambda_{<r} \} \).

We consider the two \( \Lambda_{<r} \)-semigroups \( \mathcal{A}^{\otimes r} \) to \( (\gamma S)^{\otimes r} \). We note that, by Proposition 3.3, there exists a homomorphism from \( \mathcal{A}^{\otimes r} \) to \( (\gamma S)^{\otimes r} \), which is equal to \((f^r, g)\). Note also that \( D \times X^{r-1} \in f^r(\bullet) \). Since, by Proposition 3.2, there is a homomorphism from \( (\gamma S)^{\otimes r} \) to \( \gamma(S^{\otimes r}) \), we are done by Theorem 3.1.

We have one more corollary of Theorem 3.1 and the tensor product construction. It concerns double sequences. Let \( S \) be a \( \Lambda \)-partial semigroup over \( S \) based on \( X \) A double sequence \((x_n, y_n)\) of elements of \( X \) will be called basic if the single sequence

\[
x_0, y_0, x_1, y_1, x_2, y_2, \ldots
\]

is basic. Having a basic sequence \((x_n, y_n)\), we will be interested in controlling the color on elements of the form

\[
\lambda_0(x_{m_0})\lambda'_0(y_{n_0})\lambda_1(x_{m_1})\lambda'_1(y_{n_1})\lambda_2(x_{m_2})\lambda'_2(y_{n_2})\cdots \lambda_l(x_{m_l})\lambda'_l(y_{n_l})
\]

for \( m_0 \leq n_0 < m_1 \leq n_1 < \cdots < m_l \leq n_l \) and \( \lambda_0, \lambda'_0, \ldots, \lambda_l, \lambda'_l \in \Lambda \). Let \( \mathcal{A} \) be a point based \( \Lambda \)-semigroup. For \( \lambda, \lambda' \in \Lambda \), we say that \( \lambda \) and \( \lambda' \) are conjugate if \( \lambda(\bullet) = \lambda'(\bullet) \). We say that a coloring of \( S \) is conjugate \( \mathcal{A} \)-tame on \((x_n, y_n)\) if the color of the elements of the form (3.14) such that for each \( k \leq l \)

\[
\lambda_k \text{ and } \lambda'_k \text{ are conjugate and } \lambda_k(\bullet) \lor \cdots \lor \lambda_l(\bullet) \in \Lambda(\bullet)
\]

depends only on

\[
\lambda_0(\bullet) \lor \cdots \lor \lambda_l(\bullet) \in \Lambda(\bullet).
\]
Corollary 3.5. Fix a finite set Λ. Let $S$ be a Λ-partial semigroup based on a set $X$. Let $A$ be a pointed semigroup. Let $(f, g): A \to \gamma S$ be a homomorphism. Let $v \in \gamma X$ be such that for each $\lambda \in \Lambda$

\begin{equation}
    \lambda(f(\bullet))\lambda(v) = \lambda(f(\bullet)).
\end{equation}

For $D \in f(\bullet)$ and $E \in v$ and each finite coloring of $S$, there exists a basic sequence $(x_n, y_n)$, with $x_n \in D$ and $y_n \in E$, on which the coloring is conjugate $A$-tame.

Proof. Let $A$ be based on $\bullet$. Let

$$A' = \Lambda \cup \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda : \lambda_1(\bullet) = \lambda_2(\bullet)\}.$$ 

Obviously $A' \subseteq \Lambda \times \Lambda$. Fix a point $\bullet'$. Let $A'$ be a point based $A'$-semigroup that is based on the point $(\ast, \ast')$ and the semigroup $A$ and is such that for $\lambda$, $(\lambda_1, \lambda_2) \in A'$

$$\lambda(\bullet, \bullet') = \lambda(\bullet) \quad \text{and} \quad (\lambda_1, \lambda_2)(\bullet, \bullet') = \lambda_1(\bullet).$$

Let $f': \{\bullet'\} \to \gamma X$ be given by $f'(\bullet') = v$. Since there is a homomorphism

$$(\rho, \pi): \gamma S \otimes \gamma S \to \gamma(S \otimes S)$$

and $(D, E) \in \rho \circ (f \times f')(\bullet, \bullet')$, it suffices to show that

$$(f \times f'), g): A' \to \gamma S \otimes \gamma S$$

is a homomorphism. This amounts to showing that if $\lambda, \lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1(\bullet) = \lambda_2(\bullet)$, then we have the following two equalities

\begin{equation}
    \lambda((f \times f')(\bullet, \bullet')) = g(\lambda(\bullet, \bullet')),
\end{equation}

\begin{equation}
    (\lambda_1, \lambda_2)((f \times f')(\bullet, \bullet')) = g((\lambda_1, \lambda_2)(\bullet, \bullet')).
\end{equation}

We check only the second equality, the first one being easier. To start, we note that

$$\lambda_1(f(\bullet)) = g(\lambda_1(\bullet)) = g(\lambda_2(\bullet)) = \lambda_2(f(\bullet)).$$

Using this equality and (3.15), we check the second equality in (3.16) by a direct computation as follows

\begin{align*}
    (\lambda_1, \lambda_2)((f \times f')(\bullet, \bullet')) &= (\lambda_1, \lambda_2)(f(\bullet), f'(\bullet')) = (\lambda_1, \lambda_2)(f(\bullet), v) \\
    &= \lambda_1(f(\bullet))\lambda_2(v) = \lambda_2(f(\bullet))\lambda_2(v) \\
    &= \lambda_2(f(\bullet)) = \lambda_1(f(\bullet)) = g(\lambda_1(\bullet)) \\
    &= g((\lambda_1, \lambda_2)(\bullet, \bullet')),
\end{align*}

as required. \hfill \Box

4. Monoid actions and infinitary Ramsey theorems

In this section, $M$ will be a finite monoid.

We connect Sections 2 and 3. This connection is made possible by Corollary 4.1, which translates Theorem 2.4 into a statement about the existence of a homomorphism. Ramsey theoretic consequences of this result are investigated later in the section. We introduce the notion of Ramsey monoid and we give a characterization of those among almost R-trivial monoids. We use this characterization to determine which among the monoids $I_n$ from Section 2.3 are Ramsey. This result implies
an answer to Lupini’s question [8] on possible extensions of Gowers’ theorem. We
derive some concrete Ramsey results from our general considerations. For example,
we obtain the Furstenberg–Katznelson Ramsey theorem for located words.

4.1. Connecting Theorems 2.4 and 3.1. We show how Theorem 2.4, through
Corollary 2.7, gives rise to homomorphisms needed for applications of Theorem 3.1.
In essence, we prove in Corollary 4.1 that the conclusion of Corollary 2.7 implies the
existence of a homomorphism from a point based $M$-semigroup defined from the
monoid $M$ only to an $M$-semigroup defined from an action of $M$ by endomorphisms
on a partial semigroup.

Let $S$ be a partial semigroup. For $A \subseteq S$, we say that $S$ is $A$-directed if for
all $x_1, \ldots, x_n \in S$ there exists $x \in A$ such that $x_1x, \ldots, x_nx$ are all defined. So $S$
is directed as defined in [14] if it is $S$-directed. We say that $I \subseteq S$ is a two-sided
ideal in $S$ if it is non-empty and, for $x, y \in S$ for which $xy$ is defined, $xy \in I$ if
$x \in I$ or $y \in I$.

Recall the definitions of $M$-partial semigroups from (3.1) and (3.2) in Section 3.1.
Recall also that $M$ acts by endomorphisms on the semigroup $\langle \mathcal{Y}(M) \rangle$ generated by
$\mathcal{Y}(M)$ as in Section 2.5. Denoting this action by $\beta$ and taking $y_0 \in \mathcal{Y}(M)$, we form
the point based $M$-semigroup $\langle \mathcal{Y}(M) \rangle(\beta)_{y_0}$. For the sake of simplicity, we denote
it by

$$\langle \mathcal{Y}(M) \rangle_{y_0}.$$

The following corollary will be seen to be a consequence of Corollary 2.7.

**Corollary 4.1.** Assume $M$ is almost $R$-trivial. Let $y_0 \in \mathcal{Y}(M)$ be a maximal
element, and let $M$ act on a partial semigroup $S$ by endomorphisms. Let $I \subseteq S$
be a two-sided ideal such that $S$ is $I$-directed. Then there exists a homomorphism
$(f, g): \langle \mathcal{Y}(M) \rangle_{y_0} \to \gamma(S(\alpha))$ with $I \in f(\bullet)$.

We will need a lemma.

**Lemma 4.2.** Let $S$ be a partial semigroup, and let $I$ be a two-sided ideal in $S$ such
that $S$ is $I$-directed. Then $\{U \in \gamma S: I \in U\}$ is a compact two-sided ideal in $\gamma S$.

**Proof.** For $x \in S$, let $S/x = \{y \in S: xy$ is defined\}.

Let $H = \{U \in \gamma S: I \in U\}$. Then, by definition, $H$ is clopen. It is non-empty
since, by $I$-directedness of $S$, the family $\{I\} \cup \{S/x: x \in S\}$ of subsets of $S$ has the
finite intersection property, so it is contained in an ultrafilter, which is necessarily
an element of $H$.

We check that $I \in H \ast V$ if $I \in U$ or $I \in V$. Assume first that $I \in U$. For
$x \in I$, $S/x \subseteq \{y: xy \in I\}$; therefore, since $S/x \in V$, for each $x \in I$, we have
$\{y: xy \in I\} \in V$. So $I \subseteq \{x: \{y: xy \in I\} \in V\}$. Since $I \in U$, we get

$$\{x: \{y: xy \in I\} \in V\} \in U$$

which means $I \in H \ast V$. Assume now $I \in V$. For each $x \in S$, we have $I \cap (S/x) \subseteq
\{y: xy \in I\}$. Therefore, since $I, S/x \in V$, we have $\{y: xy \in I\} \in V$ for each $x \in S$.
So

$$\{x: \{y: xy \in I\} \in V\} = S \in U,$$
which means \( I \in \mathcal{U} \ast \mathcal{V} \).

\[ \square \]

**Proof of Corollary 4.1.** We denote by \( \gamma I \) the compact two sided ideal \( \{ \mathcal{U} \in \gamma S : I \in \mathcal{U} \} \) from Lemma 4.2.

Observe that the action \( \alpha \) naturally induces an action of \( M \) by continuous endomorphisms on \( \gamma S \). We call this resulting action \( \gamma \alpha \). By Corollary 2.7, there exists a homomorphism \( g : (\mathcal{Y}(M)) \to \gamma S \) such that all maximal elements of \( \mathcal{Y}(M) \) are mapped to \( I(S) \). In particular, \( g(y_0) \in I(\gamma S) \). Since, by Lemma 4.2, \( \gamma I \) is a compact two sided ideal, we have \( I(\gamma S) \subseteq \gamma I \). Thus, \( g(y_0) \in \gamma I \), that is,

\[ I \in g(y_0). \]

Note now that if we let \( f(\bullet) = g(y_0) \), then \( (f, g) : (\mathcal{Y})y_0 \to (\gamma S)(\gamma \alpha) \) is a homomorphism. A quick check of definitions gives \( (\gamma S)(\gamma \alpha) = \gamma (S(\alpha)) \). Thus, \( (f, g) \) is as required. \[ \square \]

### 4.2. Ramsey theorems from monoids.

Given a sequence \((X_n)\) of sets, let

\[ \langle (X_n) \rangle \]

consist of all finite sequences \( x_{n_1} \cdots x_{n_l} \) for \( l \in \mathbb{N}, n_1 < \cdots < n_l \), and \( x_{n_l} \in X_{n_l} \). It will be assumed that each \( x \in X_n \) determines \( n \). In effect, we treat \( X_n \) as \( \{n\} \times X_n \).

We will write \( X_n \) instead of \( \{n\} \times X_n \) for the sake of simplicity. If \( x_1 \cdots x_k \) and \( y_1 \cdots y_l \) are elements of \( \langle (X_n) \rangle \) such that \( x_i \in X_{m_i} \), for \( m_1 < \cdots < m_k \), and \( y_t \in X_{n_t} \), for \( n_1 < \cdots < n_l \), and \( m_k < n_1 \), then we write

\[ x_1 \cdots x_k \prec y_1 \cdots y_l. \]

A **pointed** \( M \)-set is a set \( X \) equipped with an action of \( M \) and a distinguished point \( x \) such that \( Mx = X \). Let \( (X_n) \) be a sequence of pointed \( M \)-sets. The monoid \( M \) acts on \( \langle (X_n) \rangle \) in the coordinstewise manner. We say that \( (X_n) \) has the **Ramsey property** if for each finite coloring of \( \langle (X_n) \rangle \) there exist \( w_i \in \langle (X_n) \rangle \) for \( i \in \mathbb{N} \) such that
- \( w_i \prec w_{i+1} \), for each \( i \);
- each \( w_i \) contains the distinguished element of \( X_n \) as an entry;
- all words of the form
  \[ a_0(w_{i_0}) \cdots a_l(w_{i_l}), \]
  where \( l \in \mathbb{N}, a_i \in M \) with at least one \( a_i = 1_M \), are assigned the same color.

A monoid \( M \) is called **Ramsey** if each sequence of pointed \( M \)-sets has the Ramsey property.

We have the following general result.

**Theorem 4.3.** Assume \( M \) is almost R-trivial. Let \( F \) be a finite subset of \( \langle \mathcal{Y}(M) \rangle \), let \( y_0 \in \mathcal{Y}(M) \) be a maximal element of \( \mathcal{Y}(M) \), and let \( X_n \), for \( n \in \mathbb{N} \), be pointed \( M \)-sets. For each finite coloring of \( \langle (X_n) \rangle \), there exist \( w_0 \prec w_1 \prec w_2 \prec \cdots \) in \( \langle (X_n) \rangle \) such that

(i) for each \( i \), \( w_i \) contains the distinguished point of some \( X_n \) and
(ii) for each \( n_0 < \cdots < n_k \) and \( a_0, \ldots, a_k \in M \), the color of
\[
\alpha(n_0) \cdots \alpha(n_k)
\]
depends only on \( \alpha(n_0) \vee \cdots \vee \alpha(n_k) \) provided \( \alpha(n_0) \vee \cdots \vee \alpha(n_k) \in F \).

Proof. We regard \( (X_n) \) as a partial semigroup with concatenation as a partial semigroup operation and with the natural action \( \alpha \) of \( M \). This leads to the \( M \)-partial semigroup \( S(\alpha) \). Let \( I \) be the subset of \( (X_n) \) consisting of all words that contain a distinguished element of some \( X_n \). It is clear that \( I \) is a two-sided ideal and that \( (X_n) \) is \( I \)-directed. By Corollary 4.1, there exists a homomorphism
\[
(f, g): \langle \mathcal{Y}(M) \rangle_{y_0} \to \gamma S(\alpha)
\]
with \( I \in f(\bullet) \).

It is easy to find a finite set \( F' \subseteq \langle \mathcal{Y}(M) \rangle \) such that if \( z_0, \ldots, z_l \in \mathcal{Y}(M) \) and \( z_0 \vee \cdots \vee z_l \in F' \), then \( z_k \vee \cdots \vee z_l \in F' \) for each \( 0 \leq k \leq l \). Now, from the existence of the homomorphism \( (f, g) \), by Corollary 3.4, we get the existence of \( w_0 \prec w_1 \prec \cdots \in I \) such that, for \( n_0 < \cdots < n_l \) and \( a_0, \ldots, a_l \in M \), the color of \( \alpha(n_0) \cdots \alpha(n_l) \) depends only on \( \alpha(0) \vee \cdots \vee \alpha(l) \) as long as \( \alpha(0) \vee \cdots \vee \alpha(l) \in F' \), for each \( 0 \leq k \leq l \). Since this last condition is implied by \( \alpha(0) \vee \cdots \vee \alpha(l) \in F \) and since for each \( a \in M \), \( \alpha(\bullet) = \alpha(y_0) \), we are done.

We deduce from the theorem above the following result characterizing Ramsey monoids.

**Theorem 4.4.**

(i) If \( M \) is almost \( R \)-trivial and the partial order \( \mathcal{X}(M) \) is linear, then \( M \) is Ramsey.

(ii) If \( \mathcal{X}(M) \) is not linear, then the sequence of pointed \( M \)-sets \( X_n = \mathcal{X}(M) \), with the canonical action of \( M \) and with the \( R \)-class of \( 1_M \) as the distinguished point, does not have the Ramsey property.

Thus, if \( M \) is almost \( R \)-trivial, then \( M \) is Ramsey if and only if the partial order \( \mathcal{X}(M) \) is linear.

Proof. (i) Fix a sequence of pointed \( M \)-sets \( (X_n) \). We need to show that it has the Ramsey property. One checks easily that linearity of \( \mathcal{X}(M) \) implies that there exists an order preserving \( M \)-equivariant embedding of \( \mathcal{X}(M) \) to \( \mathcal{Y}(M) \) mapping the top element of \( \mathcal{X}(M) \) to a maximal element of \( \mathcal{Y}(M) \)—map the \( R \)-class of \( a \) to the set of all predecessors of the class of \( a \) in \( \mathcal{X}(M) \). We identify \( \mathcal{X}(M) \) with its image in \( \mathcal{Y}(M) \). Note that, by linearity of \( \mathcal{X}(M) \), \( \mathcal{X}(M) = \langle \mathcal{X}(M) \rangle \), so \( \mathcal{X}(M) \) is a subsemigroup of \( \langle \mathcal{Y}(M) \rangle \). Let \( y_0 \) be the top element of \( \mathcal{X}(M) \), which is the \( R \)-class of \( 1_M \). Since \( [a] \vee [1_M] = [1_M] \vee [a] = [1_M] \), for the \( R \)-class \( [a] \) of each \( a \in M \), it follows immediately from Theorem 4.3 that \( (X_n) \) has the Ramsey property. Since \( (X_n) \) was arbitrary, we get the conclusion of (i).

(ii) Let \( X_n, n \in \mathbb{N} \), be the pointed \( M \)-sets described in the statement of (ii). Let \( a, b \in M \) be two elements whose \( R \)-classes \( [a] \) and \( [b] \) are incomparable in \( \mathcal{X}(M) \). Then \( a \not\preceq bM \) and \( b \not\preceq aM \), which implies that
\[
[a] \not\preceq b\mathcal{X}(M) \quad \text{and} \quad [b] \not\preceq a\mathcal{X}(M)
\]
We color \( w \in (X_n) \) with color 0 if \([a]\) occurs in \( w \) and its first occurrence precedes all the occurrences of \([b]\), if there are any. Otherwise, we color \( w \) with color 1.
Let \( w_i \in \langle (X_n) \rangle \), \( i \in \mathbb{N} \), be such that \( w_i \prec w_{i+1} \) and with the R-class \([1_M]\) of \( 1_M \) occurring in each \( w_i \). Then, in \( a(w_0)w_1 \), \([a]\) occurs in \( a(w_0)\) and, by (4.1), \([b]\) does not occur in \( a(w_0)\). It follows that \( a(w_0)w_1 \) is assigned color 0. For similar reasons, \( b(w_0)w_1 \) is assigned color 1. Thus, the Ramsey property fails for \((X_n)\). \( \square \)

4.3. Some concrete applications. 1. Furstenberg–Katznelson’s theorem for located words. We state here the Furstenberg–Katznelson theorem for located words. The original version from [3] is stated in terms of words and is weaker. We refer the reader to [3] for the original version.

We have two finite disjoint sets \( A, B \) and \( x \not\in A \cup B \). If \( w \in \langle A \cup B \cup \{x\} \rangle \), \( v \) occurs in \( w \), and \( c \in A \cup B \cup \{x\} \), then \( w[c] \) is an element of \( \langle A \cup B \cup \{x\} \rangle \) obtained from \( w \) by replacing each occurrence of \( x \) by \( c \). If \( c_0, \ldots, c_k \in A \cup B \), let \( \overline{c_0 \cdots c_k} \) be the string obtained from \( c_0 \cdots c_k \) by removing all elements of \( B \) and then replacing each run of each \( a \in A \) by a single occurrence of \( a \).

Let \( F \) be a finite set of strings in \( A \). Color \( \langle A \cup B \rangle \) with finitely many colors. There exist \( w_0 \prec w_1 \prec w_2 \prec \cdots \) in \( \langle B \cup \{x\} \rangle \) such that \( x \) occurs in each \( w_i \) and, for each \( n_0 < \cdots < n_k \) and \( c_0, \ldots, c_k \in A \cup B \), the color of

\[ w_{n_0}[c_0] \cdots w_{n_k}[c_k] \]

depends only on \( \overline{c_0 \cdots c_k} \) provided \( \overline{c_0 \cdots c_k} \in F \).

This theorem is obtained by considering the monoid \( J(A, B) \) from Section 2.3. For brevity’s sake, set

\[ J(A, B) = M. \]

Forgetting about the Ramsey statement, we will now make some computations in \( \mathbb{Y}(M) \).

Observe that all elements of \( B \) are in the same R-class, which we denote by \( \mathbf{b} \), the R-class of each element of \( A \) consists only of this element only, and the R-class of \( 1 \) consists only of \( 1 \). So we can write

\[ \mathbb{X}(M) = \{\mathbf{b}, 1\} \cup A. \]

We have that, for each \( a \in A \),

\[ \mathbf{b} \leq \mathbb{X}(M) a \leq \mathbb{X}(M) 1 \]

and elements of \( A \) are incomparable with each other with respect to \( \leq \mathbb{X}(M) \). The action of \( M \) on \( \mathbb{X}(M) \) is induced by the action of \( M \) on itself by left multiplication.

Pick \( a_0 \in A \). Note that the sets

\[ \{\mathbf{b}\}, \{\mathbf{b}, a\}, \text{ for } a \in A, \text{ and } \{\mathbf{b}, a_0, 1\} \]

are in \( \mathbb{Y}(M) \), and we write \( \mathbf{b}, a, 1_0 \) for these elements, respectively. We notice that

\[ (4.2) \quad \mathbf{b} \leq \mathbb{Y}(M) a, \text{ for all } a \in A, \text{ and } \mathbf{b}, a_0 \leq \mathbb{Y}(M) 1_0, \]
and \( \leq_{\mathcal{Y}(M)} \) does not relate any other two of the above elements. Furthermore, \( 1_0 \) is a maximal element of \( \mathcal{Y}(M) \). The action of \( M \) on these elements is induced by the left multiplication action of \( M \) on itself, so

\[
a(1_0) = a \quad \text{and} \quad b(1_0) = b, \quad \text{for} \ a \in A, b \in B.
\]

Using relations (4.2), we observe that, for \( c_0, \ldots, c_k \in \{b\} \cup A \), the product

\[
c_0 \lor \cdots \lor c_k
\]

in the semigroup of \( \langle \mathcal{Y}(M) \rangle \) is equal to \( b \) if \( c_i = b \), for each \( i \leq k \), or is obtained from \( c_0 \lor \cdots \lor c_k \) by removing all occurrences of \( b \) and shortening a run of each \( a \in A \) to one occurrence of \( a \), if \( c_i \in A \), for some \( i \leq k \). Thus, the map assigning to a string \( c_0 \cdots c_k \) of elements of \( A \cup B \) the element \( c_0 \lor \cdots \lor c_k \) of \( \mathcal{Y}(M) \) factors through the map \( c_0 \cdots c_k \rightarrow \delta_{c_0} \cdots \delta_{c_k} \) giving an injective map \( \delta_{c_0} \cdots \delta_{c_k} \rightarrow c_0 \lor \cdots \lor c_k \).

Now we apply Theorem 4.3 to \( X_n = M \) with the action of \( M \) being left multiplication and the distinguished element being \( 1_M \). We take \( y_0 = 1_0 \) and, for the finite subset \( \langle \mathcal{Y}(M) \rangle \), we take

\[
\{c_0 \lor \cdots \lor c_k : c_0 \cdots c_k \in F\}.
\]

Now, an application of Theorem 4.3 gives a sequence \( w'_0 < w'_1 < \cdots \) in \( \langle (X_n) \rangle \). Let \( w_i \in \langle B \cup \{x\} \rangle \) be gotten from \( w'_i \) by replacing each value taken in \( A \cup \{1_M\} \) by \( x \).

It is clear that the sequence \( w'_0 < w'_1 < \cdots \) is as required.

2. Gowers’ theorem. The monoid \( G_k \) is defined in Section 2.3. Gowers’ Ramsey theorem from [4] is obtained by applying Theorem 4.3 to \( X_n = G_k \) with the left multiplication action and with the distinguished element \( 1_{G_k} \). We note that \( X(G_k) \) is linear, and we apply Theorem 4.3 as in the proof of Theorem 4.4(i).

3. The Hales–Jewett theorem for left-variable words. The Hales–Jewett theorem for located words is just the Furstenberg–Katzenelson theorem for located words with \( A = \emptyset \). To obtain the Hales–Jewett theorem for located left-variable words, see [14, Theorem 2.37], we use the monoid \( J(0, B) \) and apply the last sentence of Theorem 2.4 and Corollary 3.5.

4. Lupini’s theorem. Lupini’s Ramsey theorem from [8] is an infinitary version of a Ramsey theorem found by Bartošova and Kwiatkowska in [1]. To prove it we consider the monoid \( I_k \) defined in Section 2.3. We take for \( X_n = \{0, \ldots, k-1\} \) with the natural action of \( I_k \) and the distinguished element \( k-1 \). The result is obtained by applying Lemma 2.5 and Theorem 3.1.

4.4. The monoids \( I_n \). We consider here the monoid \( I_n \), \( n \in \mathbb{N}, n > 0 \), defined in Section 2.3. This is the monoid of all functions \( f : n \rightarrow n \) such that \( f(0) = 0 \) and \( f(i-1) \leq f(i) \leq f(i-1) + 1 \), for all \( 0 < i < n \), taken with composition. We will prove the following.

**Theorem 4.5.** The monoids \( I_n \), for \( n \geq 4 \), are not Ramsey. The monoids \( I_1, I_2, \) and \( I_3 \) are Ramsey.

We now state a theorem and question of Lupini [8] in our terminology. For \( k \in \mathbb{N} \), let \( w_k \) be a finite word in the alphabet \( \{0,1,\ldots,n-1\} \) that contains an occurrence \( n-1 \). Let \( I_n(w_k) \) be equal to the set \( \{f(w_k) : f \in I_n\} \), where \( f(w_k) \)
is the word obtained from \( w_k \) by applying \( f \) letter by letter. We take \( I_n(w_k) \) with the natural action of \( I_n \) and with \( w_k \) as the distinguished element. Note that if \( w_k \) is the word of length one whose unique letter is \( n - 1 \), then \( I_n(w_k) = n \) with the natural action of \( I_n \) on \( n \).

**Theorem** (Lupini [8]). Let \( n > 0 \), and let \( w_k = (n - 1) \). Then the sequence of pointed \( I_n \)-sets \( (I_n(w_k))_k \) has the Ramsey property.

Lupini asked the following natural question: does \( (I_n(w_k))_k \) have the Ramsey property for every choice of words \( w_k, k \in \mathbb{N} \), as above?

The following corollary to Theorem 4.5 answers this question in the negative.

**Corollary 4.6.** Let \( n \geq 4 \). For \( k \in \mathbb{N} \), let \( w_k = (01\cdots(n-1)) \). Then the sequence of pointed \( I_n \)-sets \( (I_n(w_k))_k \) does not have the Ramsey property.

**Proof.** By Theorem 4.5, \( I_n \) is not Ramsey for \( n \geq 4 \). It follows, by Theorem 4.4 and by R-triviality of \( I_n \), that the sequence \( X_k = I_n, k \in \mathbb{N} \), does not have the Ramsey property, where \( I_n \) is considered as a pointed \( I_n \)-set with the left multiplication action and with 1 as the distinguished element. Note that \( I_n \) is isomorphic as a pointed \( I_n \)-set with \( I_n(01\cdots(n-1)) \) as witnessed by the function

\[
I_n \ni f \rightarrow f(01\cdots(n-1)) \in I_n(01\cdots(n-1)).
\]

Thus, since \( w_k = 01\cdots(n-1) \), the sequence \( (I_n(w_k))_k \) does not have the Ramsey property. \( \square \)

We will give a recursive presentation of the monoid \( I_n \) that may be of some independent interest and usefulness for future applications. It will certainly make it easier for us to manipulate symbolically elements of \( I_n \) below. In the recursion, we will start with a trivial monoid and adjoin a tetris operation as in [4] at each step of the recursion.

Let \( M \) be a monoid, let \( f: M \to M \) be an endomorphism, and let \( t \in M \) be such that for all \( s \in M \) we have

\[
st = tf(s).
\]

Define

\[
\mu(M,t,f)
\]

be the triple

\[
(N,\tau,\phi),
\]

where \( N \) is a monoid, \( \tau \) is an element of \( N \), and \( \phi \) is an endomorphism of \( N \) that are obtained by the following procedure. Let \( N \) be the disjoint union of \( M \) and the set \( \{\tau s: s \in M\} \), where \( \tau \) is a new element and the expression \( \tau s \) stands for the ordered pair \( (\tau, s) \). For \( s \in M \), we write \( \tau^0 s \) for \( s \) and \( \tau^1 s \) for \( \tau s \). Define a function \( \phi: N \to M \subseteq N \) by letting, for \( s \in M \) and \( e = 0,1 \),

\[
\phi(\tau^e s) = t^e f(s),
\]
where $t^e f(s)$ is a product computed in $M$. Define multiplication on $N$ by letting, for $s_1, s_2 \in M$, and $e_1, e_2 = 0, 1$,

$$(\tau^{e_1} s_1) \cdot (\tau^{e_2} s_2) = \begin{cases} 
\tau^{e_1} (s_1 s_2), & \text{if } e_2 = 0; \\
\tau (\phi(\tau^{e_1} s_1)) s_2, & \text{if } e_2 = 1.
\end{cases}$$

where, on the right hand side, $s_1 s_2$ and $\phi(\tau^{e_1} s_1)$ are products computed in $M$.

We write $\tau$ for $\tau^{1_M}$. Note that $\tau \cdot s = \tau s$ for $s \in M$ and $\tau \cdot \tau = \tau t$. We will omit writing $\cdot$ for multiplication in $N$.

The following lemma is proved by a straightforward computation.

**Lemma 4.7.** $N$ is a monoid, $\phi$ is an endomorphism of $N$, and, for all $\sigma \in N$, we have relation (4.3), that is,

$$\sigma \tau = \tau \phi(\sigma).$$

Later, we will need the following technical lemma.

**Lemma 4.8.** For $\sigma \in N$ and $s \in M$, there exists $s' \in M$ such that $\tau s \sigma = \tau s s'$.

**Proof.** If $\sigma \in M$, then we can let $s' = \sigma$. Otherwise, $\sigma = \tau s_0$ for some $s_0 \in M$. Note that

$$\tau s \sigma = \tau s \tau s_0 = \tau t f(s) s_0 = \tau t f(s_0) = \tau t s_0,$$

and we can let $s' = t s_0$. $\square$

By recursion, we define a sequence of monoids with distinguished elements and endomorphisms. Let $M_1$ be the unique one element monoid, let $t_1$ be its unique element, and let $f_1$ be its unique endomorphism. Assume we are given a monoid $M_k$ for some $k \geq 1$ with an endomorphism $f_k$ of $M_k$ and an element $t_k$ with (4.3). Define

$$(M_{k+1}, t_{k+1}, f_{k+1}) = \mu(M_k, t_k, f_k).$$

**Proposition 4.9.** For each $k \in \mathbb{N}$, $k > 0$, $M_k$ is isomorphic to $I_k$.

**Proof.** One views $I_{k-1}$ as a submonoid of $I_k$, for $k > 1$, identifying $I_{k-1}$ with its image under the isomorphic embedding $I_{k-1} \ni s \rightarrow s' \in I_k$, where

$$s'(i) = \begin{cases} 
0, & \text{if } i = 0; \\
s(i - 1) + 1, & \text{if } 0 < i < k.
\end{cases}$$

One checks that $t_k \in I_k$ given by

$$t_k(i) = \begin{cases} 
0, & \text{if } i = 0; \\
i - 1, & \text{if } 0 < i < k.
\end{cases}$$

and $f_k : I_k \rightarrow I_k$ given by

$$f_k(s)(i) = \begin{cases} 
0, & \text{if } i = 0; \\
s(i - 1) + 1, & \text{if } 0 < i < k.
\end{cases}$$

fulfill the recursive definition of $(M_k, t_k, f_k)$. $\square$
Since, as proved in Section 2.3, $I_n$ is $R$-trivial, the partial order $\mathcal{X}(I_n)$ can be identified with $I_n$. We will make this identification and write $\leq_{I_n}$ for $\leq_{\mathcal{X}(I_n)}$. We have the following recursive formula for $\leq_{I_n}$. Obviously, $\leq_{I_1}$ is the unique partial order on the one element monoid.

**Proposition 4.10.** Let $t^{e_1}_{n+1}s_1, t^{e_2}_{n+1}s_2 \in I_{n+1}$ with $s_1, s_2 \in I_n$ and $e_1, e_2 = 0, 1$. Then $t^{e_1}_{n+1}s_1 \leq t^{e_2}_{n+1}s_2$ if and only if

$$e_2 \leq e_1 \text{ and } s_1 \leq I_n f^{e_1-e_2}_{n}(s_2).$$

**Proof.** By Proposition 4.9, we regard $(I_{n+1}, t_{n+1}, f_{n+1})$ as obtained from the triple $(I_n, t_n, f_n)$ via operation $\mu$. In particular, we regard $I_n$ as a submonoid of $I_{n+1}$. We also have $f_{n+1}(s) = f_n(s)$, for $s \in I_n$.

$(\Leftarrow)$ If $e_0 = e_1$, the implication is obvious. The remaining case is $e_2 = 0$ and $e_1 = 1$. In this case, we have $s_1 \leq I_n f_n(s_2)$, that is, $s_1 = f_n(s_2)s'$ for some $s' \in I_n$. But then

$$t_{n+1}s_1 = t_{n+1}f_n(s_2)s' = s_2(t_{n+1}s'),$$

and $t_{n+1}s_1 \leq t_{n+1}s_2$ as required.

$(\Rightarrow)$ Note that it is impossible to have $s_1 \leq I_{n+1} t_{n+1}s_2$ for $s_1, s_2 \in I_n$. Indeed, this inequality would give $s_1 = t_{n+1}s_2s$ for some $s \in I_{n+1}$, which would imply, by Lemma 4.8, that $s_1 = t_{n+1}s_2s'$ for some $s' \in I_n$. This is a contradiction since $s_2s' \in I_n$. Thus, $t^{e_1}_{n+1}s_1 \leq t^{e_2}_{n+1}s_2$ implies $e_2 \leq e_1$.

If $e_1 = e_2 = 0$, then we have $s_1 \leq I_{n+1} s_2$, which means $s_1 = s_2s$ for some $s \in I_{n+1}$. If $s \in I_n$, then $s_1 \leq I_n s_2$, as required. Otherwise, $\sigma = \tau s'$ for some $s' \in I_n$, which gives

$$s_1 = s_2\tau s' = \tau(f_n(s_2)s'),$$

which is impossible since $f_n(s_2)s' \in I_n$.

If $e_1 = e_2 = 1$, then we have $t_{n+1}s_1 \leq I_{n+1} t_{n+1}s_2$, which means $t_{n+1}s_1 = t_{n+1}s_2s$ for some $s \in I_{n+1}$. By Lemma 4.8, this equality implies $t_{n+1}s_1 = t_{n+1}(s_2s')$ for some $s' \in I_n$. Since $s_2s' \in I_n$, this equality gives $s_1 = s_2s'$, so $s_1 \leq I_n s_2$.

The last case to consider is $e_1 = 1$ and $e_2 = 0$, that is, $t_{n+1}s_1 \leq I_{n+1} s_2$. Then $t_{n+1}s_1 = s_2s$ for some $s \in I_{n+1}$. Note that $s \notin I_n$, so $s = t_{n+1}s'$ for some $s' \in I_n$. But then we have

$$t_{n+1}s_1 = s_2t_{n+1}s' = t_{n+1}f_n(s_2)s',$$

which implies $s_1 = f_n(s_2)s'$, that is, $s_1 \leq I_n f_n(s_2)$. \hfill $\square$

**Proof of Theorem 4.5.** It is easy to see from Proposition 4.10 that the orders $\leq_{I_1}$, $\leq_{I_2}$, $\leq_{I_3}$ are linear. So, by Theorem 4.4, $I_1$, $I_2$, and $I_3$ are Ramsey.

By Theorem 4.4, it remains to check that the partial order $(I_n, \leq_{I_n})$ is not linear for $n \geq 4$. By Proposition 4.9, we regard $(I_{n+1}, t_{n+1}, f_{n+1})$ as obtained from the triple $(I_n, t_n, f_n)$ via operation $\mu$. It follows from Proposition 4.10 that $\leq_{I_{n+1}}$ restricted to $I_n$ is equal to $\leq_{I_n}$. Thus, it suffices to show that $(I_4, \leq_{I_4})$ is not linear. Note that the image of $f_3$ is equal $I_2$ and $I_2$ has two elements. So there exists $s_0 \in I_3$ such that $f_3(s_0) \neq 1_{I_2}$. Thus, since $1_{I_2} = 1_{I_3}$, we get $f_3(s_0) < I_3 1_{I_3}$. 


It then follows from Proposition 4.10 that $t_4$ and $s_0$ are not comparable with respect to $\leq_I$. Indeed, $t_4 = t_1^4 t_3$ and $s_0 = t_0^4 s_0$.

Since $0 < 1$, we have $s_0 \not\leq_I t_4$; since $1 t_3 \not\leq_I f_3^{1-0}(s_0)$, we have $t_4 \not\leq_I s_0$.

\[ \square \]

The monoid $I_4$ is the first one among the monoids $I_n$, $n > 0$, that is not Ramsey. Using Propositions 4.9 and 4.10, one can compute $\leq_I$ as follows. Since $I_3$ is linearly ordered, one can list the four elements of $I_3$ as $a_3 \leq_I a_2 \leq_I a_1 \leq_I 1$. Then $I_4$ is equal to the disjoint union $I_3 \cup t_4 I_3$ and the order $\leq_I$ is the transitive closure of the relations

\begin{align*}
& a_3 \leq_I a_2 \leq_I a_1 \leq_I 1; \\
& t_4 a_3 \leq_I t_4 a_2 \leq_I t_4 a_1 \leq_I t_4; \\
& t_4 \leq_I a_1; \\
& t_4 a_1 \leq_I a_3.
\end{align*}

One can check by inspection that $M_1 = I_4 \setminus \{t_4\}$ and $M_2 = I_4 \setminus \{a_2, a_3\}$ are submonoids of $I_4$. They are R-trivial as submonoids of an R-trivial monoid [12]. One easily checks directly that $\leq_{M_1}$ and $\leq_{M_2}$ are linear, therefore, $M_1$ and $M_2$ are Ramsey by Theorem 4.4. Thus, $I_4$ is not itself Ramsey, but it is the union of two Ramsey monoids.

\section*{References}


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