1. Let $G$ be a group. \[ |G| = 45 \] Show that $G$ is solvable.

If $H$ be a Sylow $3$-subgroup in $G$. By Sylow I, \[ |H| = 9 \]

Let $k$ be the number of distinct Sylow $3$-subgroups in $G$.

By Sylow III, $k$ divides $5$ and $k \equiv 1 \mod 3$.

Therefore $k = 1$, By Sylow II, $H$ is a normal subgroup.

Since $|H| = 9$ and $|G/H| = 5$, both $H$ and $G/H$ are abelian, therefore solvable. This implies $G$ is also solvable.

2. Let $G$ be a nilpotent group. Let $H$ be a proper subgroup of $G$.

Show that $H \triangleleft N_G(H)$.

If $G$ is nilpotent, there exists $n$ such that

$G = G^{(0)} = G^{(1)} = \ldots = G^{(n)} = \{e\}$, where $G^{(i+1)} = [G^{(i)}, G]$. $G$ is a proper subgroup. There exists an integer $m$ such that $G^{(m)} \leq H$ and $G^{(m-1)} \not\leq H$. 

take $x \in G_{m(n)} \setminus H$

for any $y \in H$ \hspace{1cm} xyxy^{-1} \in G_{m(n)} \setminus H$

therefore $xyx^{-1} \in H$ \hspace{1cm} so $xHx^{-1} \subseteq H$

The same argument works for $x^{-1}$ \hspace{1cm} so $x^{-1}Hx \subseteq H$

therefore $xHx^{-1} = H$ \hspace{1cm} $x \in N_G(H)$

Since $x \in H$, this shows that $H \trianglelefteq N_G(H)$

$xHx^{-1} \subseteq H$ won't imply that $x \in N_G(H)$