The gravitational field between a mass $M$ concentrated at the point $(x, y, z)$ and a mass $m$ concentrated at the point $(x_0, y_0, z_0)$ is

$$\vec{F} = -\frac{GMm}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \left( \frac{x-x_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \hat{i} + \frac{y-y_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \hat{j} + \frac{z-z_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \hat{k} \right).$$

The gravitational potential $V$ of $\vec{F}$ is

$$V = -\frac{GMm}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}.$$ 

We have seen in class that $\vec{F} = \nabla V$. Now suppose that, instead of a point mass $M$, we have a solid region $W$ of density $\delta(x, y, z)$ and total mass $M$. The gravitational potential of $W$ acting on the point mass $m$ may be found by looking at "infinitesimal" point masses $dm = \delta(x, y, z)dV$ and adding (via integration) their individual potentials. That is, the potential of $W$ is

$$V(x_0, y_0, z_0) = -\int\int\int_W \frac{G\delta(x, y, z) dV}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}.$$ 

In the following exercises, let $W$ be the region between two concentric sphere of radii $a < b$, centered at the origin. (see the following picture.) Assume that $W$ has total mass $M$ and constant density $\delta$. The object of the following exercises is to compute the gravitational potential $V(x_0, y_0, z_0)$ of $W$ on a mass $m$ concentrated at $(x_0, y_0, z_0)$. Note that, by spherical symmetry, there is no loss of generality in taking $(x_0, y_0, z_0)$ equal to $(0, 0, r)$. So, in particular, $r$ is the distance from the point mass $m$ to the center of $W$.

![Diagram](image)

1. Show that if $r > b$, then $V(0, 0, r) = -\frac{GMm}{r}$. This is exactly the same gravitational potential as if all the mass $M$ of $W$ is concentrated at the origin. This is a key result of Newtonian mechanics. (Hint: use spherical coordinates and integrate with respect to $\varphi$ before integrating with respect to $\rho$.)

2. Show that if $r < a$, then there is no gravitational force. (Hint: Show that $V(0, 0, r)$ is actually independent of $r$. Then relate the gravitational potential to gravitational force. As in the previous question, use spherical coordinates and integrate with respect to $\varphi$ before integrating with respect to $\rho$.)

3. Find $V(0, 0, r)$ if $a < r < b$, and relate your answer with the previous questions.

**Solution.** The gravitational potential on the point mass $m$ at $(0, 0, r)$ on $z$-axis is simply given by the integral

$$V(0, 0, r) = -\int\int\int_W \frac{G\delta dV}{\sqrt{x^2 + y^2 + z^2 - 2rz + r^2}} = -\int\int\int_W \frac{G\delta dV}{\sqrt{x^2 + y^2 + z^2 - 2rz + r^2}},$$

where $G$, $m$, $\delta$ are all constants. In spherical coordinates, this is

$$V(0, 0, r) = -\int_{\theta=0}^{\theta=2\pi} \int_{\rho=a}^{\rho=b} \int_{\varphi=0}^{\varphi=\pi} \frac{G\delta}{\sqrt{\rho^2 - 2r\rho\cos\varphi + r^2}} \rho^2 \sin\varphi d\varphi d\rho d\theta.$$
In order to integrate this, we do a substitution for the inner integral. Let \( u = \rho^2 - 2r\rho \cos \varphi + r^2 \), so \( du = 2r \sin \varphi \, d\varphi \) and \( \frac{du}{2r} = \rho \sin \varphi \, d\varphi \). We have

\[
V(0, 0, r) = -\frac{Gm\delta}{2r} \int_{\theta=0}^{\varphi=\pi} \int_{\rho=a}^{\rho=b} \frac{Gm\delta}{\sqrt{u}} \frac{\rho}{2r} \, du \, d\theta
\]

1. If \( r > b \), then \( r > \rho \) for all \( \rho \in [a, b] \). And \( |\rho + r| = \rho + r \) and \( |\rho - r| = r - \rho \). So

\[
V(0, 0, r) = -\frac{Gm\delta}{r} \int_{\theta=0}^{\varphi=\pi} \int_{\rho=a}^{\rho=b} \rho \cdot \left( \rho + r - (\rho - r) \right) \, d\rho \, d\theta
\]

\[
= -\frac{Gm\delta}{r} \int_{\theta=0}^{\varphi=\pi} \int_{\rho=a}^{\rho=b} 2\rho^2 \, d\rho \, d\theta
\]

\[
= -\frac{Gm\delta}{r} \int_{\theta=0}^{\varphi=\pi} \left[ \frac{2}{3} \rho^3 \right]_{\rho=a}^{\rho=b} \, d\theta
\]

\[
= -\frac{Gm\delta}{r} \cdot \frac{2}{3} (b^3 - a^3) 2\pi
\]

\[
= -\frac{Gm\delta}{r} \cdot \frac{4}{3} \pi (b^3 - a^3)
\]

But \( \frac{4}{3} \pi (b^3 - a^3) \) is the volume of the region \( W \). So \( \delta \cdot \frac{4}{3} \pi (b^3 - a^3) = M \) and \( V(0, 0, r) = -\frac{GmM}{r} \)

2. If \( r < a \), then \( r < \rho \) for all \( \rho \in [a, b] \). And \( |\rho + r| = \rho + r \) and \( |\rho - r| = \rho - r \). So

\[
V(0, 0, r) = -\frac{Gm\delta}{r} \int_{\theta=0}^{\varphi=\pi} \int_{\rho=a}^{\rho=b} \rho \cdot \left( \rho + r - (\rho - r) \right) \, d\rho \, d\theta
\]

\[
= -\frac{Gm\delta}{r} \int_{\theta=0}^{\varphi=\pi} \int_{\rho=a}^{\rho=b} 2\rho \cdot r \, d\rho \, d\theta
\]

\[
= -Gm\delta \int_{\theta=0}^{\varphi=\pi} \left[ \rho^2 \right]_{\rho=a}^{\rho=b} \, d\theta
\]

\[
= -Gm\delta \cdot \frac{2}{3} \pi (b^3 - a^3),
\]

which is independent of \( r \) (i.e. a constant function.) Therefore, the gravity \( \vec{F}(0, 0, r) = \nabla V = (0, 0, 0) \).

3. If \( a \leq r \leq b \), then \( |\rho + r| = \rho + r \) and

\[
|\rho - r| = \begin{cases} r - \rho & a \leq \rho \leq r \\ \rho - r & r \leq \rho \leq b \end{cases}
\]
This gives us

\[ V(0, 0, r) = -\frac{Gm\delta}{r} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=a}^{\rho=b} \rho \cdot \left( |\rho + r| - |\rho - r| \right) d\rho d\theta \]

\[ = -\frac{Gm\delta}{r} \cdot 2\pi \left[ \int_{\rho=a}^{\rho=r} \rho \cdot \left( |\rho + r| - |\rho - r| \right) d\rho + \int_{\rho=r}^{\rho=b} \rho \cdot \left( |\rho + r| - |\rho - r| \right) d\rho \right] \]

\[ = -\frac{Gm\delta}{r} \cdot 2\pi \left[ \int_{\rho=a}^{\rho=r} \rho \cdot \rho + (\rho + r - (\rho - r)) d\rho + \int_{\rho=r}^{\rho=b} \rho \cdot \rho + (\rho + r - (\rho - r)) d\rho \right] \]

\[ = -\frac{Gm\delta}{r} \cdot 2\pi \left[ \int_{\rho=a}^{\rho=r} 2\rho^2 d\rho + \int_{\rho=r}^{\rho=b} 2\rho \cdot r d\rho \right] \]

\[ = -\frac{Gm\delta}{r} \cdot 2\pi \left[ \int_{\rho=a}^{\rho=r} 2\rho^2 d\rho + \int_{\rho=r}^{\rho=b} 2\rho \cdot r d\rho \right] \]

\[ = -\frac{Gm\delta}{r} \cdot 2\pi \left[ \frac{2}{3} (r^3 - a^2) + (b^2 - r^2) \cdot r \right] \]

\[ = -\frac{Gm\delta}{r} \cdot \frac{4}{3} \pi (r^3 - a^2) - Gm\delta \cdot 2\pi (b^2 - r^2) r \]