Math 241 Section E1 Worksheet 8 (Solution)
Gradient Vector Field

1. We compute the gradient vector field \( \nabla f(x, y, z) = \left(-2xe^{-(x^2+y^2+z^2)}, -2ye^{-(x^2+y^2+z^2)}, -2ze^{-(x^2+y^2+z^2)}\right) \). At the point, \( \nabla f(1, 2, 3) = \left(-2e^{-14}, -4e^{-14}, -6e^{-14}\right) \). For directional derivative, we also need to normalize the vector \( i + j + k \) and get a unit vector \( \mathbf{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \). Then \( D_u f(1, 1, 1) = \left(-2e^{-14}, -4e^{-14}, -6e^{-14}\right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\frac{12}{\sqrt{3}}e^{-14}. \)

2. Recall that the gradient vector points to the direction with the most rapid increase. \( \nabla f(x, y) = \left(-ye^x, ye^x\right) \). And hence at the point \((\pi, 0)\), the function increase most rapidly in the direction of the vector \( \nabla f(\pi, 0) = (0, -1) \).

3. (a) The surface \( x^2 - 2y^2 + 5xz = 7 \) is the level surface of the function \( F(x, y, z) = x^2 - 2y^2 + 5xy \) at level 7. Hence \( \nabla F(-1, 0, -\frac{6}{5}) \) is the normal vector to the surface \( x^2 - 2y^2 + 5xy = 7 \) at \((-1, 0, -\frac{6}{5})\). We can compute \( \nabla F(x, y, z) = (2x + 5z, -4y, 5x) \) and \( \nabla F(-1, 0, -\frac{6}{5}) = (-8, 0, -5) \). Hence the equation of the tangent plane is

\[
-8(x + 1) + 0 \cdot (y - 0) - 5(z + 6/5) = 0 \\
8x + 5z = -14
\]

(b) We can solve for \( z = \frac{7}{5x} - \frac{x}{5} - \frac{2y^2}{5x^2}, \) and \( z_x = -\frac{7}{5x^2} - \frac{1}{5} - \frac{2y^2}{5x^2} \) and \( z_y = \frac{4y}{5x} \). At \((-1, 0)\), we have \( z_x(-1, 0) = -\frac{8}{5}, \) \( z_y(-1, 0) = 0, \) and \( z(-1, 0) = -\frac{6}{5} \). So the linear approximation at \((-1, 0)\) is

\[
z = -\frac{6}{5} - \frac{8}{5}(x + 1) + 0 \cdot (y - 0).
\]