Math 241 Section E1 Worksheet 5 (Solution)
Limits and Continuity

1. (a) Along \( y = 0 \), the function \( \frac{|y|}{\sqrt{x^2 + y^2}} \) becomes \( \frac{0}{\sqrt{x^2 + 0}} = 0 \) away from the origin. Hence

\[
\lim_{(x,y) \to (0,0), y=0} \frac{|y|}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} 0 = 0.
\]

However, along \( x = 0 \), the function becomes \( \frac{|y|}{\sqrt{0 + y^2}} = \frac{|y|}{y} = 1 \) away from the origin. Hence

\[
\lim_{(x,y) \to (0,0), x=0} \frac{|y|}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} 1 = 1.
\]

Because of the discrepancy, the limit doesn’t exist.

(b) \[
\lim_{(x,y) \to (0,0)} \frac{x^2 + 2xy + y^2}{x + y} = \lim_{(x,y) \to (0,0)} \left( \frac{x + y}{x + y} \right) = \lim_{(x,y) \to (0,0)} x + y = 0.
\]

(c) \[
\lim_{(x,y) \to (0,0)} \frac{x^2 - xy}{\sqrt{x^2 - y^2}} = \lim_{(x,y) \to (0,0)} \frac{x^2 - xy}{\sqrt{x^2 - y^2}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}
\]
\[
= \lim_{(x,y) \to (0,0)} \frac{x \cdot (x - y) \cdot (\sqrt{x} + \sqrt{y})}{x - y}
\]
\[
= \lim_{(x,y) \to (0,0)} x \cdot (\sqrt{x} + \sqrt{y}) = 0
\]

(d) Notice that near \( y = 0 \), we have \( 0 \leq |\sin y| \leq y \). Hence on our function, we have the relationship

\[
0 \leq \left| \frac{x \sin^2 y}{x^2 + y^2} \right| \leq \left| \frac{x y^2}{x^2 + y^2} \right|.
\]

We have seen in the class that the right hand side has the limit \( \lim_{(x,y) \to (0,0)} \frac{x y^2}{x^2 + y^2} = 0 \). Since it’s trivial that \( \lim_{(x,y) \to (0,0)} 0 = 0 \), by the squeezing theorem, we also have the middle term \( \lim_{(x,y) \to (0,0)} x \sin^2 y = 0 \).

(e) Along the line \( x = y = 0 \) (i.e. z-axis) in \( \mathbb{R}^3 \), the function \( \frac{x y - x z - y z}{x^2 + y^2 + z^2} \) becomes \( \frac{0 - 0 - 0}{0 + 0 + z^2} = 0 \) away from the origin. Hence

\[
\lim_{(x,y,z) \to (0,0,0), x=y=0} \frac{x y - x z - y z}{x^2 + y^2 + z^2} = \lim_{(x,y,z) \to (0,0,0), x=y=0} 0 = 0.
\]

However, along the line \( \{ z = 0, x = y \} \), the function becomes \( \frac{x^2 - 0 - 0}{x^2 + x^2 + 0} = \frac{1}{2} \). Hence

\[
\lim_{(x,y,z) \to (0,0,0), z=0, x=y} \frac{x y - x z - y z}{x^2 + y^2 + z^2} = \lim_{(x,y,z) \to (0,0,0), z=0, x=y} \frac{1}{2} = \frac{1}{2}.
\]

Because of the discrepancy, the limit doesn’t exist.
2. Recall that polar coordinate records points in $\mathbb{R}^2$ by the angle $\theta$ to the $x^+$-axis in counter-clockwise direction and the distance $r$ to the origin. We can convert it to Cartesian coordinate by $x = r \cos \theta, y = r \sin \theta$. There is also a handy relationship $x^2 + y^2 = r^2$.

![Polar Coordinate System](image)

(a) 
\[
\lim_{{(x,y) \to (0,0)}} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{{r \to 0}} \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{\sqrt{r^2}} = \lim_{{r \to 0}} r(\cos^2 \theta - \sin^2 \theta) = 0
\]

(b) 
\[
\lim_{{(x,y) \to (0,0)}} \frac{x + y}{\sqrt{x^2 + y^2}} = \lim_{{r \to 0}} \frac{r \cos \theta - r \sin \theta}{\sqrt{r^2}} = \lim_{{r \to 0}} \cos \theta - \sin \theta = \cos \theta - \sin \theta.
\]

Here the limit is a function depends on $\theta$. That is to say, choosing different angles gives us different limits. Hence the limit is not well defined and hence doesn’t exist.

3. Recall that function $g(x)$ is continuous at a point $a$ if $\lim_{{x \to a}} g(x) = g(a)$. Hence, in this question, we need $\lim_{{(x,y) \to (0,0)}} g(x,y) = g(0,0) = c$. So compute

\[
\lim_{{(x,y) \to (0,0)}} g(x,y) = \lim_{{(x,y) \to (0,0)}} \frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} = \lim_{{(x,y) \to (0,0)}} \frac{x(x^2 + y^2) + (x^2 + y^2)}{x^2 + y^2} = \lim_{{(x,y) \to (0,0)}} x + 1 = 1.
\]

Hence we require $c = 0$