Math 241 Section E1 Homework 4
Solution

1. Because the triangular region is closed (it contains all its boundary points) and bounded (it doesn’t extend to infinity), the Extreme Value Theorem guarantees the existence of both the global maximum and the global minimum.

2. In the interior of $R$, we consider the gradient $\nabla f(x, y) = \langle 2(x - 2), 2(y - 1) \rangle$. Critical point happens when $\nabla f(x, y) = \langle 0, 0 \rangle$. In this case this is $(x, y) = (2, 1)$. A simple check from the picture shows that this is indeed in the interior of $R$ with critical value $f(2, 1) = 0$.

3. The boundary curve of $R$ is piece-wise smooth, so let’s label them by $L_1, L_2, L_3$ as the following.

   ![Diagram of triangle with segments L1, L2, L3]

   - Along $L_1$, $x = 1$ so we can substitute into $f(x, y)$ and get $f(1, y) = 1^2 + (y - 1)^2 = (y - 1)^2 + 1$, $y \in [-1, 3]$ as a function of 1 variable. Taking the derivative with respect to $y$, we have $\frac{d}{dy}f(1, y) = 2(y - 1)$, and critical point happens at $\frac{d}{dy}f(1, y) = 0$ (i.e. $y = 1$). So we have a critical point $(1, 1)$ along $L_1$, with critical value $f(1, 1) = 1$.
   - $L_2$ is given by the equation $2x + 5y = 17$ ($1 < x < 6$). To find critical point of $f(x, y)$ along $2x + 5y = 17$, we can use Lagrange Multiplier. Let $g_2(x, y) = 2x + 5y$. Then critical point happens when
     \[
     \begin{cases}
     \nabla f = \lambda \nabla g_2 \\
     2x + 5y = 17
     \end{cases}
     \Rightarrow
     \begin{cases}
     (2x - 4, 2y - 2) = \lambda(2, 5) \Rightarrow 2x - 4 = 2\lambda \\
     2y - 2 = 5\lambda \\
     2x + 5y = 17
     \end{cases}
     \]
     Solve the linear system, we have the critical point $(x, y) = \left( \frac{74}{29}, \frac{69}{29} \right)$ which is on the line segment $L_2$ with critical value $f \left( \frac{74}{29}, \frac{69}{29} \right) = \frac{64}{29}$.
   - $L_3$ is given by the equation $2x - 5y = 7$ ($1 < x < 6$). To find critical point of $f(x, y)$ along $2x - 5y = 7$, we can use Lagrange Multiplier. Let $g_3(x, y) = 2x - 5y$. Then critical point happens when
     \[
     \begin{cases}
     \nabla f = \lambda \nabla g_3 \\
     2x - 5y = 7
     \end{cases}
     \Rightarrow
     \begin{cases}
     (2x - 4, 2y - 2) = \lambda(2, -5) \Rightarrow 2x - 4 = 2\lambda \\
     2y - 2 = -5\lambda \\
     2x - 5y = 7
     \end{cases}
     \]
     Solve the linear system, we have the critical point $(x, y) = \left( \frac{74}{29}, -\frac{11}{29} \right)$ which is on the line segment $L_2$ with critical value $f \left( \frac{74}{29}, -\frac{11}{29} \right) = \frac{64}{29}$.
   - At the corner points $(1, 3), (1, -1), (6, 1)$, since we can’t take derivative, they are also natural critical points (derivative doesn’t exist). They have the critical values $f(1, 3) = 5, f(1, -1) = 5, f(6, 1) = 16$.

4. Now we have found 7 critical points on the region $R$. By comparing their critical value, we can conclude that the global maximum is 16 at $(6, 1)$ and the global minimum is 0 at $(2, 1)$.

5. Notice that $f(x, y)$ is the square of the distance function $\sqrt{(x - 2)^2 + (y - 1)^2}$ from $(x, y)$ to $(2, 1)$. Hence the minimum happens at the center $(2, 1)$ and the maximum happens at the farthest point $(6, 1)$.