Differential forms and vector calculus

Throughout this worksheet, all functions and differential forms are assumed infinitely differentiable.

1. Let \( \alpha = -2y^2 \, dx + 3xy \, dy + xyz \, dz \), \( \beta = \cos(x) \, dy + e^x \, dz \), and \( \omega = (2 - 2x) \, dx - 17 \, dz \). Compute the following wedge products.
   (a) \( \alpha \wedge \beta = \)
   (b) \( \alpha \wedge \omega = \)
   (c) \( \alpha \wedge \beta \wedge \omega = \)

The exterior derivative of a differential \( k \)-form \( \alpha \) on \( \mathbb{R}^n \) is a differential \( (k + 1) \)-form denoted \( d\alpha \). The definition of the exterior derivative can be deduced from the following properties that we require it satisfy.

(i.) If \( \alpha \) is a 0–form (i.e. a function), then \( d\alpha \) is the usual derivative,

\[
d\alpha = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, dx_i = f_{x_1} \, dx_1 + \ldots + f_{x_n} \, dx_n.
\]

(ii.) If \( \alpha \) is a \( k \)-form and \( \beta \) is a \( j \)-form, then \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \).

(iii.) If \( \alpha \) is any \( k \)-form, then \( d(d\alpha) = 0 \).

It turns out that these conditions uniquely determine the exterior derivative. The second condition is a kind of “graded product rule”. Combined with the first two conditions, the third condition is related to Clairaut’s Theorem.

2. For each of the following differential \( k \)-forms, compute the exterior derivative using the rules above.
   (a) The 0–form \( f = x^2 \cos(y) \) on \( \mathbb{R}^2 \).
   (b) The 1–form \( \alpha = x^2 \, dy + \sin(x) \, dz \) on \( \mathbb{R}^3 \).
   (c) The 2–form \( \beta = e^x \, dy \wedge dz + (xy + z) \, dz \wedge dx + \cos(z) \, dx \wedge dz \) on \( \mathbb{R}^3 \).
   (d) The 3–form \( \omega = x_1 x_4 \, dx_1 \wedge dx_2 \wedge dx_3 + (3x_2 + x_1) \, dx_2 \wedge dx_3 \wedge dx_4 \) on \( \mathbb{R}^4 \).

3. If \( \alpha = P \, dx + Q \, dy \) is a 1–form on \( \mathbb{R}^2 \), what is \( d\alpha \)?

4. In this exercise, you will verify that the derivatives we have considered this semester—gradient, curl, and divergence—are all encoded in the exterior derivative on \( k \)-forms on \( \mathbb{R}^3 \) (expanding on the previous exercise). First, recall from class that in \( \mathbb{R}^3 \), functions \( f : \mathbb{R}^3 \to \mathbb{R} \) are 0–forms, but can also be used to define a 3–form, by the equation \( \omega = f \, dx \wedge dy \wedge dz \). On the other hand, a vector field \( \mathbf{F} = (P, Q, R) : \mathbb{R}^3 \to \mathbb{R}^3 \) can be used define either a 1–form or a 2–form:

\[
\alpha = P \, dx + Q \, dy + R \, dz \quad \text{and} \quad \beta = P \, dy \wedge dz + Q \, dz \wedge dx + R \, dx \wedge dy.
\]

We will refer to these particular differential forms associated to these vector fields below, but also use this as general way of associating a function with a 0–form or 3–form, and a vector field with a 1–form or 2–form in the following problems.
(a) Prove that the vector field associated with the 1–form $df$ is $\nabla f$.

(b) Compute $d\alpha$ and verify that the vector field associated with this 2–form is $\text{curl} \mathbf{F} = \nabla \times \mathbf{F}$.

(c) Compute $d\beta$ and verify that the function associated with this 3–form is $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$.

(d) For a function $f$, what statement does $d(df) = 0$ translate to? What about $d(d\alpha) = 0$ for a $1$–form $\alpha$?