1. Consider the ellipsoid with implicit equation
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.
\]
(a) Parametrize this ellipsoid.

Solution. One could use the parametrization
\[
\begin{align*}
x &= a \sin \phi \cos \theta, & \quad y &= b \sin \phi \sin \theta, & \quad z &= c \cos \phi, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.
\end{align*}
\]
(b) Set up, but do not evaluate, a double integral that computes its surface area.

Solution. Since \( \mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta, b \sin \phi \sin \theta, c \cos \phi) \), one has
\[
\mathbf{r}_\phi = (a \cos \phi \cos \theta, b \cos \phi \sin \theta, -c \sin \phi), \quad \mathbf{r}_\theta = (-a \sin \phi \sin \theta, b \sin \phi \cos \theta, 0),
\]
so
\[
\mathbf{r}_\phi \times \mathbf{r}_\theta = (bc \sin^2 \phi \cos \theta, ac \sin^2 \phi \sin \theta, ab \sin \phi \cos \theta).
\]
Therefore
\[
|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{b^2 c^2 \sin^4 \phi \cos^2 \theta + a^2 c^2 \sin^4 \phi \sin^2 \theta + a^2 b^2 \sin^2 \phi \cos^2 \phi},
\]
and the surface area is computed by
\[
\begin{align*}
\text{Area} &= \int_0^{2\pi} \int_0^\pi |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^\pi \sqrt{b^2 c^2 \sin^4 \phi \cos^2 \theta + a^2 c^2 \sin^4 \phi \sin^2 \theta + a^2 b^2 \sin^2 \phi \cos^2 \phi} \, d\phi \, d\theta.
\end{align*}
\]

2. Let
\[
\mathbf{r}(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u),
\]
where \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 2\pi \).
(a) Sketch the surface parametrized by this function.

Solution. The sketch of the surface is as follows.
(b) Compute its surface area.

**Solution.** By the parametrization, one has

\[ \mathbf{r}_u = \langle -\sin u \cos v, -\sin u \sin v, \cos u \rangle, \]
\[ \mathbf{r}_v = \langle -(2 + \cos u) \sin v, (2 + \cos u) \cos v, 0 \rangle, \]

and so

\[ \mathbf{r}_u \times \mathbf{r}_v = \langle -(2 + \cos u) \cos u \cos v, -(2 + \cos u) \cos u \sin v, -(2 + \cos u) \sin u \rangle. \]

Therefore \(|\mathbf{r}_u \times \mathbf{r}_v| = 2 + \cos u\), and the surface area is computed by

\[
\text{Area} = \int_0^{2\pi} \int_0^{2\pi} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \int_0^{2\pi} \int_0^{2\pi} (2 + \cos u) \, du \, dv = 8\pi^2.
\]

3. Consider the surface integral

\[ \iint_{\Sigma} z \, dS \]

where \(\Sigma\) is the surface with sides \(S_1\) given by the cylinder \(x^2 + y^2 = 1\), \(S_2\) given by the unit disk in the \(xy\)-plane, and \(S_3\) given by the plane \(z = x + 1\). Evaluate this integral as follows:

(a) Parametrize \(S_1\) using \((\theta, z)\) coordinates.

**Solution.** One can parametrize \(S_1\) by

\[ x = \cos \theta, \ y = \sin \theta, \ z = z, \ 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq \cos \theta + 1. \]

(b) Evaluate the integral over the surface \(S_2\) without parametrizing.

**Solution.** Since \(z = 0\) on \(S_2\), we know \(\iint_{S_2} z \, dS = 0\).
(c) Parametrize $S_3$ in Cartesian coordinates and evaluate the resulting integral using polar coordinates.

**Solution.** One can parametrize $S_3$ in Cartesian coordinates

$$x = x, \quad y = y, \quad z = x + 1, \quad -1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}.$$ 

Now we move to evaluate the integral $\mathbb{I}_S z \, dS$. Obviously

$$\mathbb{I}_S z \, dS = \mathbb{I}_{S_1} z \, dS + \mathbb{I}_{S_2} z \, dS + \mathbb{I}_{S_3} z \, dS := I_1 + I_2 + I_3.$$ 

To estimate $I_1$, using the parametrization in (a), one has

$$\mathbf{r}(\theta, z) = (\cos \theta, \sin \theta, z).$$

Then

$$\mathbf{r}_\theta = (-\sin \theta, \cos \theta, 0), \quad \mathbf{r}_z = (0, 0, 1),$$

and

$$\mathbf{r}_\theta \times \mathbf{r}_z = (\cos \theta, \sin \theta, 0).$$

So $|\mathbf{r}_\theta \times \mathbf{r}_z| = 1$, and

$$I_1 = \int_0^{2\pi} \int_0^{\cos \theta + 1} z \, dz \, d\theta = \int_0^{2\pi} \frac{(\cos \theta + 1)^2}{2} \, d\theta = \frac{3\pi}{2}.$$ 

In (b) we know $I_2 = 0$. To evaluate $I_3$, by the parametrization in (c), one has

$$\mathbf{r}(x, y) = (x, y, x + 1), \quad -1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},$$

and so

$$\mathbf{r}_x = (1, 0, 1), \quad \mathbf{r}_y = (0, 1, 0), \quad \mathbf{r}_x \times \mathbf{r}_y = (-1, 0, 1).$$

Thus $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$, and the surface integral is

$$I_3 = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x + 1) \sqrt{2} \, dy \, dx = \int_{x^2 + y^2 \leq 1} (x + 1) \sqrt{2} \, dy \, dx.$$ 

To evaluate this integral, one can use the polar coordinates

$$x = r \cos \theta, \quad y = r \sin \theta, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$ 

Therefore,

$$I_3 = \int_0^{2\pi} \int_0^1 (r \cos \theta + 1) \sqrt{2} \, r \, dr \, d\theta = \sqrt{2}\pi.$$ 

Adding up all three integrals, one gets

$$\mathbb{I}_S z \, dS = I_1 + I_2 + I_3 = \frac{3\pi}{2} + \sqrt{2}\pi.$$
4. Let $C$ be the circle in the plane with equation $x^2 + y^2 - 2x = 0$.

(a) Parametrize $C$ as follows. For each choice of a slope $t$, consider the line $L_t$ whose equation is $y = tx$. Then the intersection $L_t \cap C$ of $L_t$ and $C$ contains two points, one of which is $(0, 0)$. Find the other point of intersection, and call its $x$- and $y$-coordinates $x(t)$ and $y(t)$. Compute a formula for $r(t) = (x(t), y(t))$.

**Solution.** Bring $y = tx$ into $x^2 + y^2 - 2x = 0$, then one has $x^2 + t^2x^2 - 2x = 0$, and it is easy to get $x = \frac{2}{1 + t^2}$, and then $y = \frac{2t}{1 + t^2}$. Thus $r(t) = \left(\frac{2}{1 + t^2}, \frac{2t}{1 + t^2}\right)$.

(b) Suppose that $t = \frac{p}{q}$ is a rational number. Show that $x(p/q)$ and $y(p/q)$ are also rational numbers. Explain how, by clearing denominators in $x(p/q) - 1$ and $y(p/q)$, you can find a triple of integers $U, V, W$ for which $U^2 + V^2 = W^2$.

**Solution.** Plug $t = \frac{p}{q}$ into the the parametrization, one gets

$$x(p/q) = \frac{2q^2}{p^2 + q^2}, \quad y(p/q) = \frac{2pq}{p^2 + q^2},$$

and both of them are rational numbers. Since $(x - 1)^2 + y^2 = 1$, and $x(p/q) - 1 = \frac{q^2 - p^2}{p^2 + q^2}$, then one has

$$\left(\frac{q^2 - p^2}{p^2 + q^2}\right)^2 + \left(\frac{2pq}{p^2 + q^2}\right)^2 = 1.$$

By setting

$$U = q^2 - p^2, \quad V = 2pq, \quad W = p^2 + q^2,$$

one has $U^2 + V^2 = W^2$.

(c) Compute $\int_C \frac{1}{2} (-y, x) \cdot dr$ using your parametrization above.

**Solution.** Since $r = \left(\frac{-2}{1 + t^2}, \frac{2t}{1 + t^2}\right)$, one has $r' = \left(-\frac{4t}{(1 + t^2)^2}, \frac{2 - 2t^2}{(1 + t^2)^2}\right)$. Then

$$\int_C \frac{1}{2} (-y, x) \cdot dr = \int_{-\infty}^{\infty} \frac{1}{2} \left(-\frac{2t}{1 + t^2}, \frac{2}{1 + t^2}\right) \cdot \left(-\frac{4t}{(1 + t^2)^2}, \frac{2 - 2t^2}{(1 + t^2)^2}\right) dt$$

$$= \int_{-\infty}^{\infty} \frac{2}{(1 + t^2)^2} dt = \pi.$$