\textbf{Tuesday, October 30   \hspace{1cm} Green's Theorem}

Green’s Theorem is a 2-dimensional version of the Fundamental Theorem of Calculus: it relates the (integral of) a vector field $\mathbf{F}$ on the boundary of a region $D$ to the integral of a suitable derivative of $\mathbf{F}$ over the whole of $D$.

1. Let $D$ be the unit square with vertices $(0,0), (1,0), (0,1),$ and $(1,1)$ and consider the vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x + y \rangle$. See below right for a plot.

(a) For the curve $C = \partial D$ oriented counterclockwise, directly evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Hint: to speed things up, have each group member focus on one side of $C$.

(b) Now compute $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$.

(c) Check that Green’s Theorem works in this example.

\textbf{SOLUTION:}

(a) We split the square up into four pieces, parametrizing and integrating one a time.

Right side: $C_1$ is parametrized by $\mathbf{r}_1(t) = (1, t), 0 \leq t \leq 1$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t, 1 + t) \cdot (0, 1) \, dt = \int_0^1 t + 1 \, dt = \left[ \frac{1 + t^2}{2} \right]_0^1 = \frac{3}{2}$$

Top side: $C_2$ is parametrized by $\mathbf{r}_2(t) = (1 - t, 1), 0 \leq t \leq 1$.

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (1 - t, -t) \cdot (-1, 0) \, dt = \int_0^1 t - 1 \, dt = \left[ \frac{1}{2} t^2 - t \right]_0^1 = -\frac{1}{2}$$

Left side: $C_3$ is parametrized by $\mathbf{r}_3(t) = (0, 1 - t), 0 \leq t \leq 1$.

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (0, 1 - t) \cdot (0, -1) \, dt = \int_0^1 t - 1 \, dt = \left[ \frac{1}{2} t^2 - t \right]_0^1 = -\frac{1}{2}$$

Bottom side: $C_4$ is parametrized by $\mathbf{r}_4(t) = (t, 0), 0 \leq t \leq 1$.

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (0, t) \cdot (1, 0) \, dt = \int_0^1 0 \, dt = 0$$

So, the line integral around the entire boundary $C$ going counterclockwise is

$$\int_{C=\partial D} \mathbf{F} \cdot d\mathbf{r} = \frac{3}{2} - \frac{1}{2} - \frac{1}{2} + 0 = \frac{1}{2}$$

(b)

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_0^1 \int_0^1 1 - x \, dx \, dy = \int_0^1 \left[ x - \frac{1}{2} x^2 \right]_{x=0}^{x=1} dy = \int_0^1 \frac{1}{2} \, dy = \frac{1}{2}$$
(c) The result for either computation was \( \frac{1}{2} \), demonstrating Green’s theorem for this example.

2. Compute the line integral of \( \mathbf{F}(x, y) = \langle x^3, 4x \rangle \) along the path \( C \) shown at right against a grid of unit-sized squares. To save work, use Green’s Theorem to relate this to a line integral over the vertical path joining \( B \) to \( A \). Hint: Look at the region \( D \) bounded by these two paths. Check your answer with the instructor.

**SOLUTION:**

Let \( L \) be the line segment going from \( B \) to \( A \). Then, we can now apply Green’s theorem to combination of \( C \) and \( L \). Let \( D \) be the region bounded by these two paths. Then, by Green’s theorem, since we are oriented correctly,  

\[
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \iint_D \frac{\partial}{\partial x} (4x) - \frac{\partial}{\partial y} (x^3) \, dA = \iint_D 4 \, dA = 4 \cdot \text{Area}(A) = 16
\]

because the area of the region is made of exactly 4 unit squares. The boundary of \( D \) is \( C \) and \( L \):

\[
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_L \mathbf{F} \cdot d\mathbf{r} = 16
\]

The line integral along \( L \) is easier: parametrizing \( L \) by \( \mathbf{r}(t) = (-1, t) \) for \(-1 \leq t \leq 0\), we get

\[
\int_L \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{0} (-1, -4) \cdot (0, 1) \, dt = \int_{-1}^{0} -4 \, dt = -4
\]

Putting it together,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = 16 - \int_L \mathbf{F} \cdot d\mathbf{r} = 16 - (-4) = 20
\]
3. Consider the quarter circle $C$ shown below and the vector field $\mathbf{F}(x, y) = \langle 2xe^y, x + x^2e^y \rangle$.

The goal of this problem is to compute the line integral $I_0 = \int_C \mathbf{F} \cdot d\mathbf{r}$.

(a) Parameterize $C$ and start directly expanding out $I_0$ into an ordinary integral in $t$ until you are convinced that finding $I_0$ this way will be a highly unpleasant experience.

(b) Check that $\mathbf{F}$ is not conservative, so we can’t use that trick directly to compute $I_0$.

(c) Find a function $f(x, y)$ such that $\mathbf{F} = \mathbf{G} + \nabla f$, where $\mathbf{G}$ is the vector field $\langle 0, x \rangle$.

(d) Argue geometrically that $\mathbf{G}$ integrates to 0 along any line segment contained in either the $x$-axis or the $y$-axis.

(e) Use part (d) with Green’s Theorem to show that $\int_C \mathbf{G} \cdot d\mathbf{r} = 4\pi$.

(f) Combine parts (c–e) with the Fundamental Theorem of Line Integrals to evaluate $I_0$. Check your answer with the instructor.

**SOLUTION:**

(a)

(b) Look at the partials: $rac{\partial P}{\partial y} = 2xe^y \neq 1 + 2xe^y = \frac{\partial Q}{\partial x}$. For conservative vector fields, these two partials have to be the same (since mixed partial commute).

(c) $\mathbf{F} - \mathbf{G} = \langle 2xe^y, x^2e^y \rangle$, which now is conservative, with potential function $f(x, y) = x^2 e^y$.

(d) Along the $y$-axis, $x = 0$, so $\mathbf{G} = \langle 0, 0 \rangle$. Along the $x$-axis, $\mathbf{G} = \langle 0, x \rangle$, is always exactly vertical, hence perpendicular to any portion of the $x$-axis. So, the line integral will be 0.

(e) Take $D$ to be region bounded by the arc $C$ and the two axes. By part (d),

$$\int_{\partial D} \mathbf{G} \cdot d\mathbf{r} = \int_C \mathbf{G} \cdot d\mathbf{r}$$

Now, Green’s Theorem tells us that

$$\int_{\partial D} \mathbf{G} \cdot d\mathbf{r} = \iint_D 1\, dA = \frac{1}{4}(\pi \cdot 4^2) = 4\pi$$

Combining these tells us that

$$\int_C \mathbf{G} \cdot d\mathbf{r} = 4\pi$$
(f) \( \mathbf{F} = \mathbf{G} + \nabla f \), so taking the line integrals,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{G} \cdot d\mathbf{r} + \int_C \nabla f \cdot d\mathbf{r}
\]

In (e), we computed the first integral, and we can immediately evaluate the other using the Fundamental Theorem of Calculus for Line Integrals:

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = 4\pi + (f(0,4) - f(4,0)) = 4\pi - 16
\]

4. Consider the shaded region \( V \) shown, bounded by a circle \( C_1 \) of radius 5 and two smaller circles \( C_2 \) and \( C_3 \) of radius 1. Suppose \( \mathbf{F}(x, y) = \langle P, Q \rangle \) is a vector field where \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 \) on \( V \). Assuming in addition that \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 3\pi \) and \( \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4\pi \), compute \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \). Check your answer with the instructor.

**SOLUTION:**

Note that the outer curve \( C_1 \) is oriented as we would want it to be to use for Green's theorem, but the other two are oriented backwards (walking along \( C_2 \) or \( C_3 \), our left arm points outside \( V \)). So, \( \partial V = C_1 - C_2 - C_3 \), and Green's theorem gives

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \iint_V \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 3\pi + 4\pi + 2 \cdot \text{Area}(V)
\]

The area is just that of the larger disk minus the other two:

\[
\text{Area}(V) = \pi \cdot 5^2 - \pi - \pi = 23\pi
\]

and so

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 53\pi
\]

5. Suppose \( D \) is a region in the plane bounded by a closed curve \( C \). Use Green's Theorem to show that both \( \int_C x \, dy \) and \( -\int_C y \, dx \) are equal to \( \text{Area}(D) \).

**SOLUTION:**

Using the alternate notation for line integrals, Green's theorem says

\[
\int_{\partial D} P \, dx + Q \, dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA
\]

So, applying this two the given vector fields:

\[
\int_{\partial D} x \, dy = \iint_D \frac{\partial}{\partial x} x \, dA = \iint_D 1 \, dA = \text{Area}(D)
\]

\[
-\int_{\partial D} y \, dx = -\iint_D \frac{\partial}{\partial y} y \, dA = \iint_D 1 \, dA = \text{Area}(D)
\]
6. The curve satisfying \(x^3 + y^3 = 3xy\) is called the *Folium of Descartes* and is shown at right.

(a) Let \(C\) be the “bulb” part of this folium, more precisely, the part in the positive quadrant. Show that any line \(y = tx\) for \(t > 0\) meets \(C\) in exactly two points, one of which is the origin. Use this fact to parameterize \(C\) by taking the slope \(t\) as the parameter.

(b) Use part (a) and Problem 5 to compute the area bounded by \(C\). Check your answer with the instructor.

**SOLUTION:**

(a) Our curve is defined by \(x^3 + y^3 - 3xy = 0\). If we consider a line of slope \(t > 0\) through the origin, it meets \(C\) in one other point. Let \(y = tx\); then,

\[
x^3 + t^3x^3 - 3tx^2 = 0
\]

\[
x^2(x + t^3x - 3t) = 0
\]

We can divide by \(x\) as we are looking for the solution with \(x > 0\):

\[
x = \frac{3t}{1 + t^3} \implies y = \frac{3t^2}{1 + t^3}
\]

This gives a parametrization for \(0 \leq t < \infty\).

(b) We use the first integral in the previous problem to compute the area:

\[
\text{Area}(D) = \int_C x \, dy = \int_0^\infty \frac{3t}{1 + t^3} \left( \frac{6t(1 + t^3) - 3t^2(3t^2)}{(1 + t^3)^2} \right) dt = \int_0^\infty \frac{3t^2(6 - 3t^3)}{(1 + t^3)^3} dt
\]

We can compute this integral with a \(u\)-substitution, with \(u = 1 + t^3\), \(du = 3t^2\):

\[
\int_1^\infty \frac{9 - 3u}{u^3} \, du = \int_1^\infty 9u^{-3} - 3u^{-2} \, du = \left[ \frac{-9}{2u} + 3u^{-1} \right]_{u=1}^{u=\infty} = 0 - \left( -\frac{9}{2} + 3 \right) = \frac{3}{2}
\]