1 Harmonic functions

1. Yesterday in lecture we saw that if a function \( f(z) = u(x, y) + i v(x, y) \) has a complex derivative, that the functions \( u \) and \( v \) (from \( \mathbb{R}^2 \to \mathbb{R} \)) satisfy the Cauchy-Riemann equations

\[
u_x = v_y, \quad u_y = -v_x.\]

(a) A function \( h: \mathbb{R}^2 \to \mathbb{R} \) is called harmonic if it has continuous 2nd order partial derivatives and satisfies \( h_{xx} + h_{yy} = 0 \). Use the Cauchy-Riemann equations to show that both \( u \) and \( v \) are harmonic.

(b) Let \( h: D \to \mathbb{R} \) be a harmonic function on the open unit disc. Using the 2nd derivative test, show that \( h \) cannot attain a maximum value at a point \( P \) of \( D \) unless all the second order partials of \( h \) vanish at \( P \). (In fact \( h \) must be constant, but this is more subtle)

(c) Show that \( u(x, y) = x^3 - 3xy^2 \) is a harmonic function.

(d) Find a harmonic function \( v(x, y) \) such that \( u \) and \( v \) satisfy the Cauchy-Riemann equations. What is the corresponding function \( f(z) \)?

Solutions.

(a) From the Cauchy-Riemann equations, we have \( u_{xx} = v_{yx} = v_{xy} = -u_{yy} \), so \( u \) is harmonic. Similarly, \( v_{xx} = -u_{yx} = -u_{xy} = -v_{yy} \), so \( v \) is harmonic.

(b) We apply the second derivative test to get

\[
\text{det} \begin{bmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{bmatrix} = h_{xx}h_{yy} - h_{xy}^2 = -h_{xx}^2 - h_{xy}^2 \leq 0.
\]

If this quantity is negative at a critical point \( P \), then \( h \) has a saddle at \( P \). Otherwise, \(-h_{xx}(P)^2 - h_{xy}(P)^2 = 0\), so that \( h_{xx}(P) = h_{yy}(P) = h_{xy}(P) = 0 \).

(c) Since \( u_{xx} = 6x \) and \( u_{yy} = -6x \), \( u_{xx} + u_{yy} = 0 \).

(d) The function \( v \) must satisfy

\[
\nabla v = (v_x, v_y) = (-u_y, u_x) = (6xy, 3x^3 - 3y^2).
\]

In other words, we are looking for a potential function for this vector field

\[
v = \int 6xy \, dx = 3x^2y + c_1(y),
\]

\[
v = \int (3x^3 - 3y^2) \, dy = 3x^2y - y^3 + c_2(x).
\]

Putting these together gives \( v(x, y) = 3x^2y - y^3 + c \). Then

\[
f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3) = (x + iy)^3 = z^3.
\]
2 Double integrals

This section contains material that we will go over more in the next lecture, but detailed instructions are provided to help guide you.

If $R$ is a region in the plane, and $f$ is a function $\mathbb{R}^2 \to \mathbb{R}$, then the integral $\iint_R f \, dA$ calculates the (signed) volume between surface $z = f(x, y)$ and the $xy$-plane. When $R = [a, b] \times [c, d]$ is a rectangle, this can be computed by iterating single integrals in either of the following ways:

$$\iint_R f \, dA = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy.$$

2. Evaluate the double integral as an iterated integral in both orders and check that you get the same value:

$$\iint_R xe^y \, dA$$

where $R = \{(x, y) \mid -1 \leq x \leq 2, 0 \leq y \leq 3\}$.

**Solution.** We can evaluate the integral with the bounds in either order:

$$\iint_R xe^y \, dA = \int_0^3 \int_{-1}^2 xe^y \, dy \, dx = \int_0^3 \left. \frac{1}{2}x^2e^y \right|_{x=-1}^{x=2} \, dy = \frac{3}{2} \int_0^3 e^y \, dy = \frac{3}{2} \left. e^y \right|_{y=0}^{y=3} = \frac{3}{2}(e^3 - 1),$$

or

$$\iint_R xe^y \, dA = \int_{-1}^2 \int_0^3 xe^y \, dy \, dx = \int_{-1}^2 xe^y \left|_{y=3}^{y=0} \right. \, dx = (e^3 - 1) \int_{-1}^2 x \, dx = (e^3 - 1) \left. \frac{1}{2}x^2 \right|_{x=-1}^{x=2} = \frac{3}{2}(e^3 - 1).$$

Since $xe^y$ is of the form $f(x)g(y)$ and the bounds are rectangular, we could also split the integral to begin with:

$$\iint_R xe^y \, dA = \int_{-1}^2 x \, dx \cdot \int_0^3 e^y \, dy = \frac{3}{2} \cdot (e^3 - 1).$$

3. Let $R$ be the triangular region bounded by the lines $y = x$, $x = 1$ and $y = 0$. The double integral

$$\iint_R x^2 + xy + 2 \, dA$$

represents the volume of the region $D$ between the graph $z = x^2 + xy + 2$ and the triangle $R$. The goal of this exercise is to compute this in terms of a new kind of iterated integral in which the limits of integration for the “inner” integral can depend on the “outer” variable.

(a) Sketch the triangle.

(b) For each fixed $x$ with $0 \leq x \leq 1$, we can slice through $R$ (and $D$) at that fixed $x$ value, which cuts through $R$ with a line $x =$ constant. Draw one such line through the triangle.

(c) As functions of $x \in [0, 1]$, find the $y$ coordinates of the bottom and the top of the arc of intersection of the line with $R$. 


(d) For each \( x \in [0,1] \), there is an integral that computes the area \( A(x) \) of the corresponding slice of \( D \) (the part of \( D \) that lies over the arc of \( R \)).

\[
A(x) = \int_x^1 x^2 + xy + 2 \, dy
\]

Find the limits of integration (hint: they depend on \( x \)).

(e) The volume of \( D \) is given as the integral

\[
Vol(D) = \int_R x^2 + xy + 2 \, dA = \int_0^1 A(x) \, dx.
\]

Compute the value.

(f) Repeat steps (b) - (e) reversing the roles of \( x \) and \( y \).

**Solution.**

(a) and (b):

(c) The bottom arc is \( y = 0 \) and the top arc is \( y = x \).

(d) \( A(x) = \int_0^x (x^2 + xy + 2) \, dy \)

(e) Evaluating the area function from part (d) gives

\[
A(x) = \int_0^x (x^2 + xy + 2) \, dy = \left[ x^2 y + \frac{1}{2} xy^2 + 2 y \right]_{y=0}^{y=x} = \frac{3}{2} x^3 + 2x,
\]

and so

\[
Vol(D) = \int_0^1 A(x) \, dx = \int_0^1 \left( \frac{3}{2} x^3 + 2x \right) \, dx = \left[ \frac{3}{8} x^4 + x^2 \right]_{x=0}^{x=1} = \frac{11}{8}.
\]

(f) Here the picture looks like this:
The line segment travels from the bottom arc \( x = y \) to the top arc \( x = 1 \), so our cross-sectional area function is

\[
A(y) = \int_y^1 (x^2 + xy + 2) \, dx = \left[ \frac{1}{3} x^3 + \frac{1}{2} x^2 y + 2x \right]_{x=y}^{x=1} = \frac{1}{6} (-5y^3 - 9y + 14),
\]
and the volume of \( D \) is

\[
\text{Vol}(D) = \int_0^1 A(y) \, dy = \frac{1}{6} \int_0^1 (-5y^3 - 9y + 14) \, dy = \frac{1}{6} \left[ -\frac{5}{4} y^4 - \frac{9}{2} y^2 + 14y \right]_{y=0}^{y=1} = \frac{11}{8}.
\]

4. Let \( R \) be the region between the graph \( y = 1 - x^2 \) and the \( x \)-axis in the \( xy \)-plane.

(a) Sketch the region \( R \).

**Solution:**

(b) Following the steps outlined in the previous exercise, we can find an iterated integral to compute the double integral

\[
\iint_{R} x + y \, dA = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} x + y \, dy \, dx.
\]
Decide what goes in the boxes.

**Solution:**

\[\iint_R x + y \, dA = \int_{-1}^{1} \int_{0}^{1-x^2} x + y \, dy \, dx\]

(c) Write down an iterated integral in the other order. (hint: be sure to look at the sketch of \( R \))

**Solution:**

\[\iint_R x + y \, dA = \int_{0}^{1} \int_{-\sqrt{1-y}}^{\sqrt{1-y}} x + y \, dx \, dy\]

(d) Compute the double integral using either iterated integral.

**Solution:**

\[
\begin{align*}
\iint_R x + y \, dA &= \int_{-1}^{1} \int_{0}^{1-x^2} x + y \, dy \, dx \\
&= \left[ xy + \frac{y^2}{2} \right]_{0}^{1-x^2} \, dx \\
&= \int_{-1}^{1} x(1-x^2) + \frac{(1-x^2)^2}{2} \, dx \\
&= \int_{-1}^{1} x - x^3 + \frac{1}{2} - \frac{2x^2}{2} + \frac{x^4}{2} \, dx = \frac{8}{15}
\end{align*}
\]

5. If

\[\iint_R f(x, y) \, dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y) \, dy \, dx\]

what is the region \( R \)?

**Solution:** A circle with center (0,0) and radius 2: