1 Harmonic functions

1. Yesterday in lecture we saw that if a function $f(z) = u(x, y) + i v(x, y)$ has a complex derivative, that the functions $u$ and $v$ (from $\mathbb{R}^2 \to \mathbb{R}$) satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

(a) A function $h: \mathbb{R}^2 \to \mathbb{R}$ is called harmonic if it has continuous 2nd order partial derivatives and satisfies $h_{xx} + h_{yy} = 0$. Use the Cauchy-Riemann equations to show that both $u$ and $v$ are harmonic.

(b) Let $h: D \to \mathbb{R}$ be a harmonic function on the open unit disc. Using the 2nd derivative test, show that $h$ cannot attain a maximum value at a point $P$ of $D$ unless all the second order partials of $h$ vanish at $P$. (In fact $h$ must be constant, but this is harder to prove)

(c) Show that $u(x, y) = x^3 - 3xy^2$ is a harmonic function.

(d) Find a harmonic function $v(x, y)$ such that $u$ and $v$ satisfy the Cauchy-Riemann equations. What is the corresponding function $f(z)$?

2 Double integrals

This section contains material that we will go over more in the next lecture, but detailed instructions are provided to help guide you.

If $R$ is a region in the plane, and $f$ is a function $\mathbb{R}^2 \to \mathbb{R}$, then the integral $\iint_R f \, dA$ calculates the (signed) volume between surface $z = f(x, y)$ and the $xy$-plane. When $R = [a, b] \times [c, d]$ is a rectangle, this can be computed by iterating single integrals in either of the following ways:

$$\iint_R f \, dA = \int_a^b \left( \int_c^d f(x, y) \, dy \right) dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy.$$

2. Evaluate the double integral as an iterated integral in both orders and check that you get the same value:

$$\iint_R xe^y \, dA$$

where $R = \{(x, y) \mid -1 \leq x \leq 2, 0 \leq y \leq 3\}$. 
3. Let $R$ be the triangular region bounded by the lines $y = x$, $x = 1$ and $y = 0$. The double integral

$$\iint_R x^2 + xy + 2 \, dA$$

represents the volume of the region $D$ between the graph $z = x^2 + xy + 2$ and the triangle $R$. The goal of this exercise is to compute this in terms of a new kind of iterated integral in which the limits of integration for the “inner” integral can depend on the “outer” variable.

(a) Sketch the triangle.

(b) For each fixed $x$ with $0 \leq x \leq 1$, we can slice through $R$ (and $D$) at that fixed $x$ value, which cuts through $R$ with a line $x = \text{constant}$. Draw one such line through the triangle.

(c) As functions of $x \in [0,1]$, find the $y$ coordinates of the bottom and the top of the arc of intersection of the line with $R$.

(d) For each $x \in [0,1]$, there is an integral that computes the area $A(x)$ of the corresponding slice of $D$ (the part of $D$ that lies over the arc of $R$).

$$A(x) = \int \int \ldots$$

Find the limits of integration (hint: they depend on $x$).

(e) The volume of $D$ is given as the integral

$$Vol(D) = \iint_R x^2 + xy + 2 \, dA = \int_0^1 A(x) \, dx.$$ 

Compute the value.

(f) Repeat steps (b) - (e) reversing the roles of $x$ and $y$.

4. Let $R$ be the region between the graph $y = 1 - x^2$ and the $x$–axis in the $xy$-plane.

(a) Sketch the region $R$.

(b) Following the steps outlined in the previous exercise, we can find an iterated integral to compute the double integral

$$\iint_R x + y \, dA = \int_1^0 \int \ldots$$

Decide what goes in the boxes.

(c) Write down an iterated integral in the other order. (hint: be sure to look at the sketch of $R$)

(d) Compute the double integral using either iterated integral.

5. If

$$\iint_R f(x,y) \, dA = \int_2^{-2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y) \, dy \, dx$$

what is the region $R$?