Thursday, September 6  **  Functions of several variables; Limits.

1. For each of the following functions \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \), draw a sketch of the graph together with pictures of some level sets.

   (a) \( f(x, y) = xy \)

   (b) \( f(x) = |x| \). Please note here that \( x \) is a vector. In coordinates, this function is \( f(x, y) = \sqrt{x^2 + y^2} \).

2. Consider the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) given by

\[
   f(x, y) = \frac{2x^3 y}{x^6 + y^2} \quad \text{for} \quad (x, y) \neq 0
\]

In this problem, you’ll consider \( \lim_{(x,y) \to 0} f(x, y) \).

   (a) Look at the values of \( f \) on the \( x \)- and \( y \)-axes. What do these values show the limit \( \lim_{(x,y) \to 0} f(x, y) \) must be if it exists?

   (b) Show that along each line in \( \mathbb{R}^2 \) through the origin, the limit of \( f \) exists and is 0.

   (c) Despite this, show that the limit \( \lim_{(x,y) \to 0} f(x, y) \) does not exist by finding a curve over which \( f \) takes on the constant value 1.

3. Consider the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) given by

\[
   f(x, y) = \frac{xy^2}{\sqrt{x^2 + y^2}} \quad \text{for} \quad (x, y) \neq 0
\]

In this problem, you will show that \( \lim_{h \to 0} f(h) = 0 \).

   (a) Let \( h = (x, y) \). Write down \( |h| \) in terms of \( x \) and \( y \). What is the relationship between \( |h| \) and \( |x| \) and \( |y| \)?

   (b) Using part (a), find an upper bound for \( |f(h)| \) which depends only on \( |h| \).

   (c) How small must \( |h| \) be in order for your bound to be less than a given \( \epsilon > 0 \)?

   (d) Put all the previous steps together to give a proof that \( \lim_{h \to 0} f(h) = 0 \). That is, given an arbitrary \( \epsilon > 0 \), find a \( \delta > 0 \) so that when \( 0 < |h| < \delta \) we have \( |f(h)| < \epsilon \).

   (e) Explain why limit laws mentioned in class on Wednesday aren’t enough to compute this particular limit.
More problems on projections, distances, and planes (if there is extra time).

4. Let \( \mathbf{a} = \mathbf{i} + \mathbf{j} \) and \( \mathbf{b} = 2\mathbf{i} - \mathbf{j} \).

   (a) Calculate \( \text{proj}_b \mathbf{a} \) and draw a picture of it together with \( \mathbf{a} \) and \( \mathbf{b} \).

   (b) The orthogonal complement of the vector \( \mathbf{a} \) with respect to \( \mathbf{b} \) is defined by
   
   \[
   \text{orth}_b \mathbf{a} = \mathbf{a} - \text{proj}_b \mathbf{a}.
   \]

   Calculate \( \text{orth}_b \mathbf{a} \) and draw two copies of it in your picture from part (a), one based at \( \mathbf{0} \) and the other at \( \text{proj}_b \mathbf{a} \).

   (c) Check that \( \text{orth}_b \mathbf{a} \) calculated in (b) is orthogonal to \( \text{proj}_b \mathbf{a} \) calculated in (a).

   (d) Find the distance of the point \((1, 1)\) from the line \((x, y) = t(2, -1)\). Hint: relate this to your picture.

5. Let \( \mathbf{a} \) and \( \mathbf{b} \) be vectors in \( \mathbb{R}^n \). Use the definitions of \( \text{proj}_b \mathbf{a} \) and \( \text{orth}_b \mathbf{a} \) to show that \( \text{orth}_b \mathbf{a} \) is always orthogonal to \( \text{proj}_b \mathbf{a} \).

6. Find the distance between the point \( P(3, 4, -1) \) and the line \( \mathbf{l}(t) = (2, 3, -2) + t(1, -1, 1) \). Hint: Consider a vector starting at some point on the line and ending at \( P \), and connect this to what you learned in Problem 1.

7. What is the angle between the line \( \mathbf{l}(t) = (1 + 2t, 2 - 4t, 7 - 2t) \) and the plane \( x + 2y - 3z = 6 \)? This is not something that was defined in class; you might want to discuss with your group what this should mean.

8. The Triangle Inequality. Let \( \mathbf{a} \) and \( \mathbf{b} \) be any vectors in \( \mathbb{R}^n \). The triangle inequality states that \( |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}| \).

   (a) Give a geometric interpretation of the triangle inequality. (E.g. draw a picture in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) that represents this inequality.)

   (b) Use what we know about the dot product to explain why \( |\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}| \). This is called the Cauchy-Schwarz inequality.

   (c) Use part (b) to justify the triangle inequality. Hint: Start with the fact that \( |\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \) and then use properties of the dot product and the Cauchy-Schwarz inequality to manipulate the right-hand side into looking like \( |\mathbf{a}|^2 + |\mathbf{b}|^2 \).