One-day excursion: Holomorphic functions and Cauchy’s Integral Theorem

October 8, 2018

[[Standard disclaimer: These notes are not a replacement for lecture. They may be incomplete, and do not contain the pictures I draw on the board.]]

1 The complex plane

Introduce complex numbers \( \mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R} \} \). The points \( z = x + iy \) on the complex plane correspond to points \( (x, y) \) in \( \mathbb{R}^2 \). Addition of complex numbers corresponds to vector addition in \( \mathbb{R}^2 \):

\[
(x + iy) + (u + iv) = (x + u) + i(y + v),
\]

while multiplication is something new/special:

\[
(x + iy)(u + iv) = xu + iyu + ixv = i^2 yv = (xu - yv) + i(yu + xv).
\]

Points in \( \mathbb{C} \) can be represented using polar coordinates as \( z = x + iy = r \cos \theta + ir \sin \theta \). Use angle addition formulas to show how the product of complex numbers is gotten by multiplying the \( r \)'s and adding the \( \theta \)'s.

Introduce norm \( |z| = \sqrt{x^2 + y^2} \), the conjugate \( \bar{z} = x - iy \). Since \( z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \), nonzero complex numbers have multiplicative inverse \( z^{-1} = \frac{\bar{z}}{|z|^2} \).

2 The complex derivative

Let \( U \subset \mathbb{C} \) be an open set (use identification of \( \mathbb{C} \) with \( \mathbb{R}^2 \)) and let \( f : U \to \mathbb{C} \) be a function. Then we can write

\[
f(x + iy) = u(x, y) + iv(x, y)
\]

where \( u, v \) are real valued functions, the “real and imaginary parts” of \( f \).

**Definition:** \( f \) is (complex) differentiable at \( z \) if the following limit exists:

\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}.
\]
If so, we call this $f'(z)$. If $f'$ exists and is continuous on $U$, we say that $f$ is holomorphic or analytic (much stronger than “real differentiable”).

Take time to notice how this definition compares to the definition of differentiability from Calculus 1, as well as our definition of differentiability for functions $\mathbb{R}^2 \to \mathbb{R}^2$. In particular, note that $h = h_1 + ih_2$ is a complex number, and

$$f(z + h) - f(z) = \frac{h}{|h|^2} (f(z + h) - f(z))$$

$$= \frac{h_1(u(x + h_1, y + h_2) - u(x, y)) - h_2(v(x + h_1, y + h_2) - v(x, y))}{h_1^2 + h_2^2}$$

$$+ i \frac{h_1(v(x + h_1, y + h_2) - v(x, y)) + h_2(u(x + h_1, y + h_2) - u(x, y))}{h_1^2 + h_2^2},$$

and we are taking the limit of this as $(h_1, h_2) \to (0, 0)$. This is the same kind of 2-variable limit we have been dealing with in this class! We have one for the real part and one for the imaginary part.

Suppose that $f: U \to \mathbb{C}$ is holomorphic. We examine the limit above as we let $h \in \mathbb{C}$ approach zero along the two axes. Taking $h$ of the form $h_1 + 0i$, we get

$$f'(z) = \lim_{h_1 \to 0} \frac{u(x + h_1, y + h_2) - u(x, y) + i(v(x + h_1, y + h_2))}{h_1}$$

$$= u_x(x, y) + iv_x(x, y).$$

Similarly, letting $h \to 0$ along the imaginary axis yields

$$f'(z) = v_y(x, y) - iu_y(x, y).$$

These are supposed to be equal to one another! In particular, we can equate their real and imaginary parts to get

$$y_x = v_y, \quad u_y = -v_x.$$

These are called the Cauchy-Riemann Equations.

### 3 Complex line integrals

We can define line integrals for complex functions, by

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C udx - vdy + i\int_C vdx + udy.$$

Alternatively, with a parameterization $r(t) = r_1(t) + ir_2(t)$ from $[a, b] \to \mathbb{C}$, this can be written as

$$\int_C f(z)dz = \int_a^b f(r(t))r'(t)dt.$$

Note that this the product $f(r(t))r'(t)$ is complex multiplication, not the dot product.

The real and imaginary parts of this line integral are the integrals of the vector fields $\langle u, -v \rangle$ and $\langle v, u \rangle$ on $U \subset \mathbb{R}^2$. If $f$ is holomorphic and $U$ is simply connected, then the Cauchy-Riemann equations imply that these vector fields are both conservative.
Theorem 1 (Cauchy’s Integral Formula). Let $f: U \to \mathbb{C}$ be holomorphic on a simply connected open set $U$. Then for any closed curve $C \subset U$, 

$$\int_C f(z)dz = 0.$$ 

Moreover, the following is also true.

Theorem 2 (Cauchy’s Integral Formula). Let $f: U \to \mathbb{C}$ be holomorphic on a simply connected open set $U$. Let $C \subset U$ be a simple closed curve, oriented counterclockwise. Then for any point $a$ inside of $C$, 

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z-a}.$$ 

In fact, $f$ is infinitely differentiable and 

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^{n+1}}.$$