Isoperimetric inequalities for eigenvalues of triangles

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19 September 2007
Outline

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   - New results

2 Symmetrization techniques
   - Steiner symmetrization
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   - Polarization

3 Proofs
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   - Isosceles triangles
   - Circular sectors
   - Monotonicity
**Eigenvalues**

Let $D$ be an open set. Eigenvalues $\lambda_i$ of the Dirichlet Laplacian on $D$ will be called eigenvalues of $D$. They form a nondecreasing sequence such that $0 < \lambda_1 < \lambda_2$. 
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### Geometric quantities

- $A$ - area of $D$
- $R$ - inradius
- $L$ - perimeter
- $d$ - diameter
- $\lambda_D = \lambda_1$ - first eigenvalue of $D$

### Triangles

- $h$ - altitude perpendicular to the longest side
- $\gamma$ - smallest angle
**Classical isoperimetric inequality**

Among all domains with fixed area $A$, the ball minimizes perimeter $L$.

$$L^2 \geq 4\pi A.$$
Classical isoperimetric inequality
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Theorem (Faber-Krahn inequality)
Among all domains with fixed area $A$, the ball minimizes the first eigenvalue.

$$\lambda_{P(n)} A \geq \lambda_{R(n)} A.$$ 

(Proved for triangles and quadrilaterals.)
**Classical isoperimetric inequality**

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**Theorem (Faber-Krahn inequality)**

Among all domains with fixed area $A$, the ball minimizes the first eigenvalue.

**Pólya’s isoperimetric conjecture**

Among all polygons $P(n)$ with $n$ sides and fixed area $A$, the regular polygon $R(n)$ minimizes the first eigenvalue.

$$\lambda_{P(n)}A \geq \lambda_{R(n)}A.$$ 

(Proved for triangles and quadrilaterals.)
Known eigenvalues

- ball
- rectangles
- annuli
- circular sectors
- equilateral triangle
- right triangles with angles $\pi/4$ or $\pi/6$
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Methods of obtaining lower bounds

- Domain monotonicity (larger domain $\rightarrow$ smaller eigenvalue)
- Restricting to a subdomain (right isosceles triangle is a half of a square)
- Special analytical cases
- Symmetrization
**Theorem (Freitas ’06)**

*For arbitrary triangle $T$*

$$
\lambda_T \geq \pi^2 \left( \frac{4}{d^2} + \frac{d^2}{4A^2} \right).
$$
Theorem (Freitas ’06)

For arbitrary triangle $T$

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Theorem

$$\lambda_T \geq \pi^2 \left( \frac{4}{d^2 + h^2} + \frac{d^2 + h^2}{4A^2} \right).$$
**Theorem (Freitas ’06)**

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**Theorem**

Let $T$ be a triangle with fixed area $A$ and smallest angle $\gamma$. Then the eigenvalue $\lambda_T$ is decreasing with diameter $d$. 
Theorem

The eigenvalue $\lambda_T$ of an isosceles triangle $T$ with area $A$ and smallest angle $\gamma$ is bigger than the eigenvalue of a sector with the same area and angle.
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**Theorem**

Given fixed area $A$, the eigenvalue of an isosceles triangle decreases when the smallest angle $\gamma$ increases.
**Theorem**

The eigenvalue $\lambda_T$ of an isosceles triangle $T$ with area $A$ and smallest angle $\gamma$ is bigger than the eigenvalue of a sector with the same area and angle.

**Theorem**

Given fixed area $A$, the eigenvalue of an isosceles triangle decreases when the smallest angle $\gamma$ increases. The same is true for right triangles.
**Definition of Steiner symmetrization**

Fix a line \( l \) and domain \( D \).

**Action on triangles**

- Preserves area
- Decreases perimeter
- Decreases the first eigenvalue
Definition of Steiner symmetrization

Fix a line $I$ and domain $D$. Consider cross-sections of a domain $D$ perpendicular to $I$.

Action on triangles

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**Definition of Steiner symmetrization**

Fix a line $I$ and domain $D$. Consider cross-sections of a domain $D$ perpendicular to $I$. The Steiner symmetrization of $D$ is the domain $D^*$ formed by the cross-sections centered around $I$.

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Definition of Steiner symmetrization

Fix a line $l$ and domain $D$. Consider cross-sections of a domain $D$ perpendicular to $l$. The Steiner symmetrization of $D$ is the domain $D^*$ formed by the cross-sections centered around $l$.

Action on triangles

- preserves area
- decreases perimeter
- decreases the first eigenvalue
**Definition of continuous Steiner symmetrization**

Let \( 0 \leq t \leq 1 \) be a time parameter. Let \( D^0 \) equal to initial domain \( D \), and \( D^1 \) equal to the Steiner symmetrization \( D^* \).

**Action on triangles**

\[
D = D^0 \\
D^1 = D^*
\]
Definition of continuous Steiner symmetrization

Let $0 \leq t \leq 1$ be a time parameter. Let $D^0$ equal to initial domain $D$, and $D^1$ equal to the Steiner symmetrization $D^*$. For $0 < t < 1$ we define a continuous Steiner symmetrization as a domain formed by the cross-sections at time $t$ (partially shifted proportionally to $t$).

Action on triangles

$D = D^0 \rightarrow D^t \rightarrow D^1 = D^*$
Definition of continuous Steiner symmetrization

Let $0 \leq t \leq 1$ be a time parameter. Let $D^0$ equal to initial domain $D$, and $D^1$ equal to the Steiner symmetrization $D^*$. For $0 < t < 1$ we define a continuous Steiner symmetrization as a domain formed by the cross-sections at time $t$ (partially shifted proportionally to $t$).

Action on triangles

$D = D^0$ \hspace{1cm} $D^t$ \hspace{1cm} $D^1 = D^*$

As $t$ increases:
- area remains fixed
- perimeter decreases
- the first eigenvalue decreases
Definition of polarization

Fix a domain $D$ and a line $l$. This line splits the space into two half-spaces $H_1$ and $H_2$. Let $D'$ be a reflection of $D$ about $l$. The polarization $D^p$ of a set $D$ consists of $D \cap H_1$, $D' \cap H_1$ and any point from $(D \cup D') \cap H_2$ such that its reflection is already included.

Action on triangles

$D$

$D'$

$l$
Definition of polarization

Fix a domain $D$ and a line $l$. This line splits the space into two half-spaces $H_1$ and $H_2$. Let $D'$ be a reflection of $D$ about $l$. The polarization $D^p$ of a set $D$ consists of $D \cap H_1$, $D' \cap H_1$ and any point from $(D \cup D') \cap H_2$ such that its reflection is already included.

Action on triangles

$D$, $D'$, $D^p$, $l$
**Definition of polarization**

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**Action on triangles**

- preserves area
- decreases perimeter
- decreases the first eigenvalue
**Alternative proof of Freitas’s bound**

**Steps**

1. Steiner symmetrization with respect to $l_1$. 
Alternative proof of Freitas’s bound

Steps

2. Polarization with respect to $l_2$. 

$A$ $B$ $C$

$\alpha$

$l_2$
Alternative proof of Freitas’s bound

Steps

2. Polarization with respect to $l_2$. 

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Alternative proof of Freitas’s bound

Steps
3. Polarization with respect to $l_3$. 
Alternative proof of Freitas’s bound

Steps

3. Polarization with respect to $l_3$. 
Alternative proof of Freitas's bound

Steps

4. The same procedure on the other side
Steps

5. \( \lambda_T \geq \lambda_R = \pi^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right) = \pi^2 \left( \frac{4}{d^2} + \frac{d^2}{4A^2} \right) \)
Improved lower bound using rectangles

**Steps**
- Start with an already symmetrized triangle.
Improved lower bound using rectangles

Steps

- Start with an already symmetrized triangle.
- Steiner symmetrization with respect to the longest side.
Improved lower bound using rectangles

Steps
- Start with an already symmetrized triangle.
- Steiner symmetrization with respect to the longest side.
- Steiner symmetrization with respect to the altitude.
Improved lower bound using rectangles

\[ \lambda_T \geq \lambda_R = \pi^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right) = \pi^2 \left( \frac{4}{d^2 + h^2} + \frac{d^2 + h^2}{4A^2} \right) \]

Steps

- Start with an already symmetrized triangle.
- Steiner symmetrization with respect to the longest side.
- Steiner symmetrization with respect to the altitude.
We want to fit a gray triangle with area $A$ inside a red one with area $A + \varepsilon$. Then by the scaling property we can take a limit $\varepsilon \to 0$. This shows that given area $A$ and the smallest angle $\gamma$, the eigenvalue decreases with diameter.
Symmetrization into isosceles triangles

We need to define a sequence of polarizations with respect to certain bisectors.
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Symmetrization into isosceles triangles

The reversed sequence of reflections gives a valid sequence of polarizations.
Symmetrization into circular sector
Symmetrization into circular sector
Symmetrization into circular sector
Symmetrization into circular sector
Symmetrization into circular sector
Proofs

Circular sectors

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Monotonicity for isosceles triangles
Monotonicity for isosceles triangles

Continuous Steiner symmetrization with respect to $l_1$. 
Monotonicity for isosceles triangles

- Continuous Steiner symmetrization with respect to $l_1$.
- Continuous Steiner symmetrization with respect to $l_2$. 