There are 9 pages and 6 questions, for a total of 90 points.

No calculators, no books.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit.

Good luck!

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>90</td>
</tr>
<tr>
<td>Score:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Useful formulas:

\[ y = y_e + y_p \]

\[ \omega_0 = \sqrt{\frac{k}{m}} \]

\[ p = c \sqrt{\frac{2m}{c^2 m}} \]

\[ \omega_1 = \sqrt{\omega_0^2 - p^2} \]

\[ y_p = y_1 u_1 + y_2 u_2 \]

\[ \begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases} \]
1. Write down the form of the complementary solution $y_c$ and the particular solution $y_p$
by the method of undetermined coefficients. You do not need to evaluate any
coefficients.

(a) (5 points)

$$ y'' + 2y' + y = x \sin 2x. $$

The roots are $r_1 = -1$ and $r_2 = -1$. Hence

$$ y_c = (a + bx)e^{-x}. $$

The particular solution is

$$ y_p = (c + dx) \sin 2x + (e + fx) \cos 2x. $$

Note that $(c + dx)(\sin x + e \cos x)$ is not a good guess. $\sin x$ and $\cos x$ must have
different linear terms in front of them. We need 4 constants.

(b) (5 points)

$$ y'' + y = x \sin x. $$

Here the roots are $r_1 = i$ and $r_2 = -i$. Hence the complementary solution is

$$ y_c = a \sin x + b \cos x. $$

Our first guess for the particular solution is

$$ y_p = (c + dx) \sin x + (e + fx) \cos x. $$

But this gives duplication, hence we multiply by $x$. We get

$$ y_p = x((c + dx) \sin x + (e + fx) \cos x). $$

Note that we cannot avoid the part of the solution with $\sin x$. This would be ok
if there was no duplication. If you take derivative of $x \cos x$ you will get $\sin x$ and
$\cos x$. 
2. Do #2 or #3, but not both. We have the following damped equation

\[ x''(t) + x'(t) + kx(t) = 0, \]

where \( k > 1/2 \) is a constant.

(a) (5 points) Show that this equation is underdamped, then write down the general solution and evaluate the pseudo-period \( T (= 2\pi/\text{pseudo-frequency}) \).

The characteristic polynomial is

\[ r^2 + r + k = 0. \]

Hence the roots are

\[ r = \frac{-1 \pm \sqrt{1 - 4k}}{2}. \]

Notice that \( 1 - 4k \) is negative \( (k > 1/2) \), hence we have complex roots. This also means that equation is underdamped and it has the solution

\[ x = e^{(-1/2)t} \cos \left( \frac{\sqrt{4k - 1}}{2} t - \gamma \right). \]

The pseudo-frequency is \( \sqrt{4k - 1/2} \) and the pseudo-period equals

\[ T = \frac{2\pi}{\sqrt{4k - 1/2}}. \]

(b) (10 points) Find a value of \( k \) such that the amplitude of oscillations is reduced by 50% over each pseudo-period of time.

The amplitude envelope of the solution is \( e^{(-1/2)t} \). We want to find \( T \) such that

\[ 50%e^{(-1/2)t} = e^{(-1/2)(t+T)}. \]

This gives \( e^{-T/2} = 1/2 \), or \( e^{T/2} = 2 \). Taking logarithms, \( T = 2 \ln 2 \). Using part (a) we get

\[ \frac{2\pi}{\sqrt{4k - 1/2}} = 2 \ln 2. \]

This gives

\[ k = \frac{1}{4} \left( \frac{2\pi}{\ln 2} \right)^2 + \frac{1}{4}. \]
3. (15 points) Do #2 or #3, but not both. Find the general solution of

\[ y'' + \frac{2}{x}y' = -\frac{\sin x}{x}. \]

*Hint:* Try \( y = x^s \) as a solution of the homogeneous part and find possible values of \( s \). Then use variation of parameters.

This equation has nonconstant coefficients, hence method of undetermined coefficients will not work. There is also no guess for negative powers of \( x \).

First we look for 2 solutions of the homogeneous part. After multiplying by \( x^2 \) we get

\[ x^2y'' + 2xy' = 0. \]

We guess \( y = x^s \). This gives \( y' = sx^{s-1}, \ y'' = s(s-1)x^{s-2} \). We substitute this into equation to get

\[ x^2s(s-1)x^{s-2} + 2xsx^{s-1} = 0. \]

This gives \( s(s - 1) + 2s = 0 \), or \( s(s + 1) = 0 \). We get 2 solutions

\[ y_1 = x^0 = 1, \quad y_2 = x^{-1}. \]

Write the system of equations

\[
\begin{align*}
1u_1' + x^{-1}u_2' &= 0, \\
0u_1' - x^{-2}u_2' &= -\frac{\sin x}{x}.
\end{align*}
\]

Solve to get

\[ y = a + bx^{-1} + \frac{\sin x}{x}. \]
4. (10 points) Find the general solution of

\[ \begin{align*}
    x'_1 &= 9x_1 + 5x_2 \\
    x'_2 &= -6x_1 - 2x_2
\end{align*} \]

We need to find eigenvalues and eigenvectors.

\[ 0 = \det(A - \lambda I) = \begin{bmatrix} 9 - \lambda & 5 \\ -6 & -2 - \lambda \end{bmatrix} = \lambda^2 - 7\lambda + 12. \]

Hence \( \lambda_1 = 3 \), \( \lambda_2 = 4 \).

\( \lambda_1 = 3 \). We substitute \( \lambda = 3 \) and solve \((A - 3I)\vec{v} = 0\).

\[ \begin{bmatrix} 6 & 5 & 0 \\ -6 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 6 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Hence \( 6v_1 + 5v_2 = 0 \). This gives

\[ \vec{v}_1 = \begin{bmatrix} -5 \\ 6 \end{bmatrix} \]

\( \lambda_1 = 4 \). We substitute \( \lambda = 4 \) and solve \((A - 4I)\vec{v} = 0\).

\[ \begin{bmatrix} 5 & 5 & 0 \\ -6 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Hence \( v_1 + v_2 = 0 \). This gives

\[ \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

Therefore the general solution is

\[ \vec{x} = c_1 \begin{bmatrix} -5 \\ 6 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}. \]
5. (a) (10 points) Solve the system

\[
\begin{align*}
m' &= -aw \\
w' &= bm
\end{align*}
\]

Again, we look for eigenvalues.

\[
0 = \det(A - \lambda I) = \begin{bmatrix} -\lambda & -a \\ b & -\lambda \end{bmatrix} = \lambda^2 + ab.
\]

Hence eigenvalues are \( \lambda = \pm i \sqrt{ab} \). Take \( \lambda = i \sqrt{ab} \). We have

\[
\begin{bmatrix}
-i \sqrt{ab} & -a & 0 \\
0 & -i \sqrt{ab} & 0
\end{bmatrix}
\sim
\begin{bmatrix}
-i \sqrt{b} & -\sqrt{a} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

You just need to make sure that the second row is all zeros. The first can be in any form. This means that you can get different eigenvectors. We get

\[\vec{v} = \begin{bmatrix} i \sqrt{a} \\ \sqrt{b} \end{bmatrix}.\]

We get a complex solution

\[
\begin{bmatrix}
i \sqrt{a} \\ \sqrt{b}
\end{bmatrix} e^{i \sqrt{ab}t} = \begin{bmatrix} i \sqrt{a} \\ \sqrt{b}
\end{bmatrix} (\cos(\sqrt{ab}t) + i \sin(\sqrt{ab}t))
\]

\[
= \begin{bmatrix} -\sqrt{a} \sin(\sqrt{ab}t) \\ \sqrt{b} \cos(\sqrt{ab}t) \end{bmatrix} + i \begin{bmatrix} \sqrt{a} \cos(\sqrt{ab}t) \\ \sqrt{b} \sin(\sqrt{ab}t) \end{bmatrix}
\]

We separated real and imaginary parts of the solution. Let \( \vec{x}_1 \) be the real part and \( \vec{x}_2 \) the imaginary part (with dropped \( i \)). The general solution now equals

\[
\begin{bmatrix}
m \\
w
\end{bmatrix} = c_1 \vec{x}_1 + c_2 \vec{x}_2.
\]
(b) (5 points) Sketch the phase portrait and state what type it is.
   Eigenvalues are purely imaginary, hence we should get a center. Trajectories form
closed loops around the fixed point at the origin.

(c) (5 points) Assume

\[
\begin{cases}
  m(t) = \text{(number of moose on an island)} - 100, \\
  w(t) = \text{(number of wolves on the island)} - 10.
\end{cases}
\]

Take a typical trajectory on the phase portrait and interpret in terms of the moose
and wolf populations.
6. (a) (15 points) Use the matrix exponential method to solve the system

\[
\begin{align*}
x' &= 4x + 6y \\
y' &= \frac{5}{2}x + 2y
\end{align*}
\]

with initial condition \(x(0) = -1, y(0) = 1\).

The solution is

\[
\begin{bmatrix} x \\ y \end{bmatrix} = e^{At} \begin{bmatrix} -1 \\ 1 \end{bmatrix},
\]

where \(A = \begin{bmatrix} 4 & 6 \\ \frac{5}{2} & 2 \end{bmatrix}\). To find the matrix exponential we first find eigenvalues and eigenvectors. We get \(\lambda_1 = -1, \lambda_2 = 7\) and eigenvectors

\[
\vec{v}_1 = \begin{bmatrix} -6 \\ 5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]

Now we define

\[
E = [\vec{v}_1, \vec{v}_2] = \begin{bmatrix} -6 & 2 \\ 5 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}.
\]

We also have

\[
E^{-1} = \frac{1}{-16} \begin{bmatrix} 1 & -2 \\ -5 & -6 \end{bmatrix}, \quad e^{Dt} = \begin{bmatrix} e^{-1t} & 0 \\ 0 & e^{-7t} \end{bmatrix}.
\]

We know that matrix exponential can be computed by

\[
e^{At} = e^{EDE^{-1}} = E e^{Dt} E^{-1}
\]

\[
= \frac{1}{-16} \begin{bmatrix} -6e^{-t} - 10e^{7t} & 12e^{-t} + 12e^{7t} \\ 5e^{-t} - 5e^{7t} & -10e^{-t} - 6e^{7t} \end{bmatrix}.
\]

The last expression is obtained by multiplying three matrices. Finally,

\[
\begin{bmatrix} x \\ y \end{bmatrix} = e^{At} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} -18e^{-t} + 2e^{7t} \\ 15e^{-t} + e^{7t} \end{bmatrix}
\]
(b) (5 points) Sketch the phase portrait and state what type it is.

One eigenvalue is positive, the other is negative. Hence we should get a saddle. Trajectories point toward the origin in the direction of the eigenvector belonging to $\lambda = -1$, and away from origin in the direction of the eigenvector belonging to $\lambda = 7$. 