Final Exam Review Problems.

December 4, 2015

1. For the first order linear equation:
   \[ y' + p(x)y = q(x) \]
   
   (a) Derive the formula for integrating factor.
   
   (b) Using the integrating factor solve the following equation:
       \[ y' + \frac{1}{x}y = 2 \]

2. Solve the initial value problem:
   \[ x(x + 1)y' - y = 2xy \]
   \[ y(1) = 1 \]

3. Section 1.6: # 20, 28, 29, 32, 37

4. Find the general solution of ODE:
   \[ yy'' + (y')^2 = y(y')^3 \]

5. Find all eigenvalues and associated eigenfunctions for the following boundary value problem for \( y(x) \):
   \[ y'' - 2y' + \lambda y = 0 \]
   \[ y(0) = y(2) = 0 \]

6. Find the general solution of this forced mechanical oscillator. What will happen to the solution as \( t \to +\infty \)? Does this result depend on initial conditions and why?
   \[ x'' + 2x' + 7x = 2\sin 3t \]
7. Find the general solution of ODE:

\[ y'' - 6y' + 8y = 8x^2 + 1 + 2e^{4x} \]

8. Use method of variation of parameters to find particular solution of ODEs:

\[ y'' + 4y = \tan x \]
\[ y'' - 2y' + y = e^x \]

9. Given harmonic oscillator:

\[ x'' + 2\delta x' + \omega_0^2 x = f_0 \sin \omega t \]

a) Derive the amplification factor \( \rho(\omega) \) for undamped harmonic oscillator.
b) Derive the amplification factor \( \rho(\omega) \) for damped harmonic oscillator and determine the forcing frequency leading to oscillations of largest amplitude.

10. Given one solution of the equation determine the general solution of ODE:

\[ 6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4y = 0 \]

where \( y_1(x) = \cos 2x \).

11. Given the first order system:

\[ \begin{align*}
  y' &= -x \\
  x' &= y
\end{align*} \]

a) Find the solution. b) Determining trajectories and sketch phase plane portrait.

12. Given boundary value problem (BVP):

\[ \begin{align*}
  x'' + x &= f(t) \\
  x(0) &= x(2) = 0
\end{align*} \]

(a) Find Fourier sine series for the function:

\[ f(t) = \begin{cases} 
  t & \text{for } 0 \leq t \leq 1 \\
  1 & \text{for } 1 \leq t \leq 2
\end{cases} \]

(b) Find the point in \( 0 \leq t \leq 2 \) for which \( f(t) \) is \textbf{not} equal to its Fourier sine series.
(c) Using $f(t)$ defined in (a) find the solution of inhomogeneous BVP in terms of its Fourier series.

13. Consider the inhomogeneous linear system:

\[
\begin{align*}
    x_1' &= 2x_1 + x_2 + f_1(t) \\
    x_2' &= 2x_2 + f_2(t)
\end{align*}
\]

(a) Find the matrix of coefficients $A$ associated with homogeneous system.

(b) Find the matrix exponential $e^{At}$ and use it to solve the homogeneous IVP:

\[x(0) = (1, -1)^T.\]

(c) Find the solution of the inhomogeneous IVP with:

\[x(0) = (1, -1)^T, \quad \text{and} \quad f = (0, 2t)^T\]

14. Consider homogeneous system:

\[
\begin{align*}
    x_1' &= 3x_1 - x_2 + 2x_3 \\
    x_2' &= -x_1 + 2x_2 - x_3 \\
    x_3' &= -2x_1 + x_2 - x_3
\end{align*}
\]

(a) Write the matrix of coefficients $A$ associated with this linear system.

(b) Find the eigenvalues $\lambda$ of $A$ with their algebraic multiplicities.

(c) Find the eigenvectors corresponding to eigenvalues in part (b) and geometric multiplicities of eigenvalues.

(d) Identify defective eigenvalue, find the defect and find corresponding Jordan chain.

(e) Find the fundamental matrix of solutions for $x' = Ax$.

15. Consider homogeneous system:

\[
\begin{align*}
    x_1' &= 3x_1 + x_2 + x_3 \\
    x_2' &= -5x_1 - 3x_2 - x_3 \\
    x_3' &= 5x_1 + 5x_2 + 3x_3
\end{align*}
\]
(a) Find the eigenvalues $\lambda$ of $A$ with their algebraic multiplicities.

(b) Find the eigenvectors corresponding to eigenvalues in part (a).

(c) Find all linearly independent solutions.

**16.** Consider homogeneous system:

$$
\begin{align*}
x_1' &= 3x_1 + x_3 \\
x_2' &= 9x_1 - x_2 + 2x_3 \\
x_3' &= -9x_1 + 4x_2 - x_3
\end{align*}
$$

(a) Find the eigenvalues $\lambda$ of $A$ with their algebraic multiplicities.

(b) Find the eigenvectors corresponding to eigenvalues in part (a).

(c) Solve the IVP:

$$
x(0) = (0, 0, 17)^T
$$

**17.** Consider homogeneous system:

$$
\begin{align*}
x_1' &= 7x_1 - 5x_2 \\
x_2' &= 4x_1 + 3x_2
\end{align*}
$$

(a) Find the eigenvalues $\lambda$ of $A$ with their algebraic multiplicities.

(b) Find the eigenvectors corresponding to eigenvalues in part (a).

(c) Sketch the phase plane portrait, mark with arrows how trajectory is traversed as $t$ increases.

**18.** Consider homogeneous system:

$$
\begin{align*}
x_1' &= 6x_1 - 7x_2 \\
x_2' &= x_1 - 2x_2
\end{align*}
$$

(a) Find the eigenvalues $\lambda$ of $A$ with their algebraic multiplicities.

(b) Find the eigenvectors corresponding to eigenvalues in part (a).

(c) Sketch the phase plane portrait, mark with arrows how trajectory is traversed as $t$ increases.
19. Consider homogeneous system:

\[
\begin{align*}
    x'_1 &= -3x_1 + 5x_2 - 5x_3 \\
    x'_2 &= 3x_1 - x_2 + 3x_3 \\
    x'_3 &= 8x_1 - 8x_2 + 10x_3
\end{align*}
\]

(a) Find the eigenvalues $\lambda$ of $A$ with their algebraic multiplicities.
(b) Find the eigenvectors corresponding to eigenvalues in part (a).
(c) Find defect of eigenvalue and generalized eigenvectors.
(d) Find all linearly independent solutions of the linear system.

20. Given heat equation:

\[ u_t = 2u_{xx} \]

(a) Solve the boundary value problem (BVP):

\[
\begin{align*}
    0 < x < 1; \ t > 0; \ u(0, t) &= u(1, t) = 0; \\
    u(x, 0) &= 5\sin \pi x - \frac{1}{5}\sin 3\pi x
\end{align*}
\]

(b) Solve the boundary value problem (BVP):

\[
\begin{align*}
    0 < x < 100; \ t > 0; \ u(0, t) &= u(100, t) = 0; \\
    u(x, 0) &= x(100 - x)
\end{align*}
\]

(c) Repeat part (b) with boundary conditions $u_x(0, t) = u(100, t) = 0$.
(d) Repeat part (b) with boundary conditions $u(0, t) = u_x(100, t) = 0$.

21. Given wave equation:

\[ u_{tt} = 25u_{xx} \]

Solve the boundary value problem (BVP):

\[
\begin{align*}
    0 < x < \pi; \ t > 0; \ u(0, t) &= u(\pi, t) = 0; \\
    u(x, 0) &= \frac{1}{4}\sin \pi x; \quad u_t(x, 0) = 10\sin 2\pi x
\end{align*}
\]
22. Given wave equation:

\[ u_{tt} = 4u_{xx} \]

Solve the boundary value problem (BVP):

\[ 0 < x < \pi; \ t > 0; \ u(0, t) = u(\pi, t) = 0; \]
\[ u(x, 0) = \sin x; \quad u_t(x, 0) = 1 \]

23. Given Laplace equation in a semiinfinite strip \( 0 < x < \pi \) and \( y > 0 \):

\[ u_{xx} + u_{yy} = 0 \]

Solve the boundary value problem (BVP):

\[ u(0, y) = u(\pi, y) = 0 \]
\[ u(x, 0) = x(\pi - x) \]
\[ \lim_{y \to \infty} u(x, y) \to 0 \]

23. Given Laplace equation in the upper half of a unit disk \( 0 < r < 1 \) and \( 0 < \phi < \pi \):

\[ u_{xx} + u_{yy} = 0 \]

Solve the boundary value problem (BVP):

\[ u(r, 0) = u(r, \pi) = 0 \]
\[ u(1, \phi) = \phi(\pi - \phi) \]