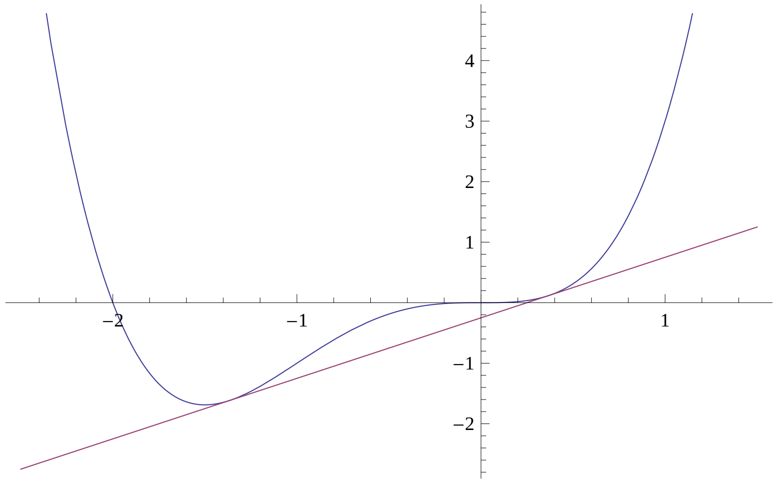
Problem:
Find the bitanget to the graph of $f(x) = x^4 + 2x^3$.



Solution 1:
Let the bitangent be tangent at $(x_1, f(x_1))$ and $(x_2, f(x_2))$. So

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1) = f'(x_2)$$

This is equivalent to the system of equations

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'(x_1) = 0$$
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'(x_2) = 0$$

This is equivalent to the symetrical system of equations (the equations don't change when x_1 and x_2 are interchanged)

$$\left\{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'(x_1)\right\} + \left\{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'(x_2)\right\} = 0$$
$$\left\{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'(x_1)\right\} - \left\{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'(x_2)\right\} = 0$$

Due to the special symmetry of these equations, these factor, respectively, as

$$2(x_1 - x_2)^2 (x_1 + x_2 + 1) = 0$$
$$2(x_1 - x_2)^1 \left(2x_1^2 + 2x_1 x_2 + 2x_2^2 + 3x_1 + 3x_2\right) = 0$$

Since $x_1 \neq x_2$,

$$1 + x_1 + x_2 = 0$$
$$3x_1 + 2x_1^2 + 3x_2 + 2x_1 x_2 + 2x_2^2 = 0$$

or after some algebra, equvalently,

$$x_1 + x_2 = -1$$
$$2x_1 x_2 = -1$$

Until now, all of the equations have been symmetric in x_1 and x_2. A solution to the above system is

$$x_{1,2} = \frac{-1 \pm \sqrt{3}}{2}$$

The tanget line is given by either of the two equations

$$y - f(x_1) = f'(x_1)(x - x_1)$$
$$y - f(x_2) = f'(x_2)(x - x_2)$$

Note that if the sign on the $\sqrt{3}$ is fliped these two equations are interchanged. Since these two equations represent the same tangent line, the $\sqrt{3}$ in the final formula must vanish. By plugging everything in, the bitangent line is found to be

$$y = x - \frac{1}{4}$$

Solution 2:
The equation of an arbitary tangent is $y = f'(a)(x - a) + f(a)$. For most values of a this tangent line will intersect the curve $y = f(x)$ at either 0 or 2 other points. Only for two special values of a will this tanget line interesect $y = f(x)$ at only one other point and this makes it a bitangent. We can determine a by looking at the solution to the equation $f(x) - f'(a)(x - a) - f(a) = 0$ with respect to x. This is a degree four equation in x, which when expanded becomes

$$x^4 + 2x^3 + \left(-4a^3 - 6a^2\right)x + 4a^3 + 3a^4 = 0$$

We know $x = a$ is a double root of this equation; taking out the factor $(x - a)^2$ gives

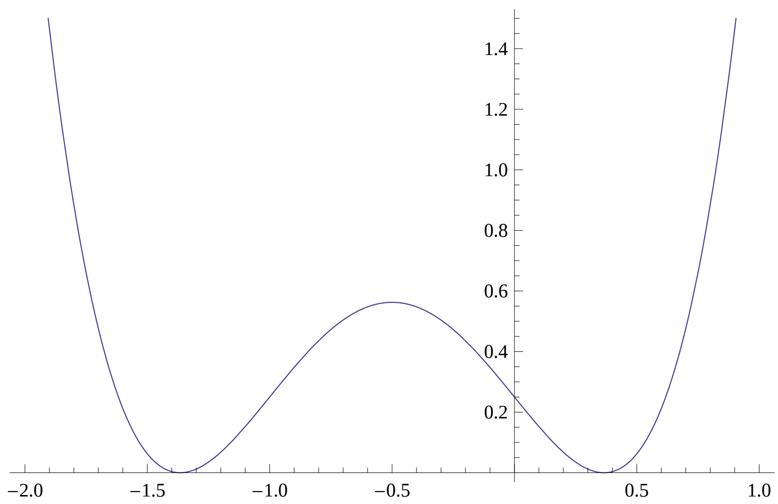
$$(x - a)^2\left(x^2 + (2a + 2)x + 3a^2 + 4a\right) = 0$$

On order for this equation to have exactly two solutions, as required by the bitangent, the discriminant of the quadratic factor must vansh:

$$(2a + 2)^2 - 4(1)\left(3a^2 + 4a\right) = 0 \qquad \Rightarrow a = \frac{-1 \pm \sqrt{3}}{2}$$

Agrees with Solution 1.

Solution 3:
Let the equation of the bitangent be $y = m x + c$. Consider what the graph of $x^4 + 2x^3 - (m x + c)$ must look like.



It must have two double roots in order that $y = m\,x + c$ is a bitangent . So

$$x^4 + 2\,x^3 - m\,x - c = (x - r_1)^2\,(x - r_2)^2 = \left(x^2 + a\,x + b\right)^2$$

for some a and b. By expanding the right hand side and equating coefficients on like powers of x,

$$
\begin{array}{lll}
x^4: & 1 = 1 & \\
x^3: & 2 = 2\,a & \Rightarrow a = 1 \\
x^2: & 0 = a^2 + 2\,b & \Rightarrow b = -1/2 \\
x^1: & -m = 2\,a\,b & \Rightarrow m = 1 \\
x^0: & -c = b^2 & \Rightarrow c = -1/4
\end{array}
$$