MODULAR EQUATIONS FOR PICARD MODULAR FUNCTIONS

DANIEL SCHULTZ

ABSTRACT. Modular equations for Appell’s $F_1$ is developed from the work of Koike et al. A simpler derivation of their modular equations for Picard modular functions is found, and further modular equation are derived.

1. Introduction

In [14] the striking identity

\[(1.1) \quad F \left(1 - x^3, 1 - y^3\right) = \frac{3}{1 + x + y} F \left(\frac{1 + \omega x + \bar{\omega} y}{1 + x + y}, \frac{1 + \bar{\omega} x + \omega y}{1 + x + y}\right)\]

was derived in connection with the common limit of a three term iteration. Here, $\omega = e^{2\pi i / 3}$, $F(x, y) = F_1 \left(\frac{1}{3}; \frac{1}{3}; 1; x, y\right)$, and $F_1$ is the first of the four two-variable hypergeometric functions first introduced by Appell [4]. We will show that such an identity is part a larger class of identities and derive the next member of this class,

\[(1.2) \quad F \left(\frac{x^3(y^2 + 3)(xy^2 - 3x - 6y)}{(xy - 3)^3(x + 3)}, \frac{y^3(x^2 + 3)(y^2 - 3y - 6x)}{(xy - 3)^3(x + 3)}\right) = \frac{xy - 3}{xy - 3x - 3y - 3} \times F \left(\frac{(x^2 + 3)(y^2 - 3y - 6x)}{(xy + 3)(xy - 3x - 3y - 3)^3}, \frac{(y^2 + 3)(x^2 - 3x - 3y)}{(xy + 3)(xy - 3x - 3y - 3)^3}\right).

The $F_1$ function is defined for $|x| < 1$ and $|y| < 1$ by

\[(1.3) \quad F_1 \left(a; b_1; b_2; c; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!},\]

where $(a)_n = a(a+1)\cdots(a+n-1)$, and an analytic continuation for Re$(a) > 0$ and Re$(c-a) > 0$ is given by the integral representation

\[(1.4) \quad F_1 \left(a; b_1; b_2; c; x, y\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2} dt.

Due to the reduction formula

\[F_1(a; b_1, b_2; c; x, x) = 2F_1(a, b_1 + b_2; c; x)\]

the specialization of $x = y$ in (1.1) and (1.2) reduces them to transformations involving the one-variable function

\[F(x) = 2F_1 \left(\frac{1}{3}; \frac{2}{3}; 1; x\right)\].

Thus, (1.1) and (1.2) can be viewed as two-variable generalizations of modular equations arising from Ramanujan’s theory of elliptic functions to base three (theory of signature three). For this reason, we first review several of the key results of this theory in Section 2 before stating the main results on two-variable generalizations in Section 3. Section 4 introduces six $\Theta$ functions of two variables that invert the function $F(x, y)$, while Sections 5 and 6 are devoted to developing machinery to give mechanical proofs of (1.1) and (1.2).
Finally, Section 7 gives motivated and simple proofs of (1.1) and (1.2) based on identities of \( \Theta \) functions, although one set of \( \Theta \) function identities is still unproven.

2. Definition of the One-Variable Cubic Modular Equations and Statements of Known Results

As in the introduction, set

\[
F(x) := F(x, x) = 2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; x \right).
\]

The defining relation of a cubic modular equation is

**Definition 2.1.** The variables \( m, \alpha, \) and \( \beta \) are said to be related by a one-variable cubic modular equation of degree \( n \in \mathbb{N} \) when the simultaneous relations

\[
\begin{align*}
F(\beta)m &= F(\alpha) \\
F(1 - \beta)m &= nF(1 - \alpha)
\end{align*}
\]

hold.

The Borwein \( \Theta \) functions are defined as (set \( q = e^{2\pi i \tau} \)):

\[
\begin{align*}
a(\tau) &= \sum_{\mu \in \mathbb{Z}[\omega]} q^{\mu \bar{\mu}}, \\
c(\tau) &= \sum_{\mu \in \mathbb{Z}[\omega] + \frac{2 + \omega}{3}} q^{\mu \bar{\mu}}, \\
b(\tau) &= \sum_{\mu \in \mathbb{Z}[\omega]} \omega^{\mu + \bar{\mu}} q^{\mu \bar{\mu}}.
\end{align*}
\]

**Proposition 2.2** (Catalog of Borwein \( \Theta \) function evaluations). Suppose that \( m, \alpha, \) and \( \beta \) are related by a one-variable cubic modular equation of degree \( n \). We then have the following table for converting identities among Borwein \( \Theta \) functions to modular equations and vice-versa.

\[
\begin{align*}
a(\tau) &= z, & a(n\tau) &= m^{-1}z \\
c(\tau) &= \alpha^{1/3}z, & c(n\tau) &= m^{-1}\beta^{1/3}z, \\
b(\tau) &= (1 - \alpha)^{1/3}z, & b(n\tau) &= m^{-1}(1 - \beta)^{1/3}z,
\end{align*}
\]

where

\[
z = F(\alpha)
\]

It is known that a one-variable cubic modular equation on degree \( n \) induces a algebraic relation between \( \alpha \) and \( \beta \) of degree exactly

\[
\frac{3n}{4} \prod_{\substack{p \mid 3n \text{ prime}}} \left( 1 + \frac{1}{p} \right).
\]

The following result on cubic modular equations of degree 3 is equivalent to Theorem 7.4 in [3].

**Theorem 2.3.** The following is a parameterization of the one-variable cubic modular equation of degree 3.

\[
\begin{align*}
\beta &= x^3, \\
\alpha &= 1 - \left( \frac{1 - x}{1 + 2x} \right)^3, \\
m &= 1 + 2x.
\end{align*}
\]
The following result on cubic modular equations of degree 4 is equivalent to Theorem 6.4 in [3], where the parameter $p = \frac{2x}{3y}$ was used.

**Theorem 2.4.** The following is a parameterization of the one-variable cubic modular equation of degree 4.

$$
\beta = \frac{x^4(x^2 - 9)}{(x^2 - 3)^3}, \\
\alpha = \frac{x(x + 3)^3(x^2 - 9)}{(x^2 - 6x - 3)^3}, \\
m = \frac{x^2 - 6x - 3}{x^2 - 3}.
$$

3. Definition of the Two-Variable Modular Equations and Statements of Main Results

For certain rational values of the parameters $a, b_1, b_2, c$, the Schwarz map associated with the $F_1$ function may be inverted by automorphic functions, and identities such as (1.1) and (1.2) provide modular equations for these automorphic functions. The series in (1.3) satisfies a system of partial differential equations given by

$$
x(1 - x)f_{xx} = ab_1 f + \frac{b_1(1 - y)y}{x - y} f_y + \left(a + b_1 + 1\right)x - c - \frac{b_2(1 - x)y}{x - y} f_x,
$$

$$
y(1 - y)f_{yy} = ab_2 f + \frac{b_2(1 - x)x}{y - x} f_x + \left(a + b_2 + 1\right)y - c - \frac{b_1x(1 - y)}{y - x} f_y,
$$

$$
(x - y)f_{xy} = b_2 f_x - b_1 f_y.
$$

In fact, for general values of the parameters $a, b_1, b_2, c$, $F_1(x, y)$ is the unique solution that is holomorphic at $(0, 0)$ and takes the value 1 there. At any point in the complement of the singular locus $\Lambda = \{(x, y) : xy(1 - x)(1 - y)(x - y) \neq 0\}$ there is a basis of three holomorphic solutions. Let $\eta_0, \eta_1, \eta_2$ denote such a basis, and let $\Lambda'$ denote a maximal simply connected subset of $\Lambda$ on which $\eta_0, \eta_1$ and $\eta_2$ are single-valued. A loop $\pi$ in the fundamental group of $\Lambda$ acts on this basis of solutions as an element $\theta_\pi \in \text{GL}(3, \mathbb{C})$, and this is the monodromy representation of the $F_1$ system. One important property that the $F_1$ system of differential equations should possess is that the Schwarz map $\Phi : \Lambda' \to \mathbb{C}P^2$ defined by

$$(x, y) \mapsto [\eta_0(x, y) : \eta_1(x, y) : \eta_2(x, y)]$$

must be invertible in a global sense. That is, together with the point $(x, y)$ let us consider a loop $\pi \in \pi_1(\Lambda)$. Then, we need the map

$$(x, y, \pi) \mapsto [\theta_\pi \cdot (\eta_0 \ \eta_1 \ \eta_2)^T]$$

to be a bijection, considering two loops to be the same when their associated monodromy matrices are projectively equivalent. This is essentially a condition for the discreteness of the monodromy group, and the condition given in [6, p. 66] is that, provided all of the numbers $1 - a, 1 + a - c, b_1, b_2, c - b_1 - b_2$ are in the interval $(0, 1)$, all of the numbers

$$
1 - b_1 - b_2, \ a - b_1, \ b_1 - c + 1, \ c - a - b_1, \ b_1 + b_2 - a, \ c - 1, \ a - b_2, \ b_2 - c + 1, \ c - a - b_2, \ b_1 + b_2 + a - c
$$
must either be reciprocals of integers or zero. In what follows, we will set $a = b_1 = b_2 = \frac{1}{3}$, $c = 1$, and $\omega = e^{2\pi i/3}$ and consider the basis of solutions
\[
\eta_0(x, y) = F(x, y), \\
\eta_1(x, y) = F(1 - x, 1 - y), \\
\eta_2(x, y) = x^{-1/3}F\left(\frac{y}{x}, \frac{1}{x}\right) - y^{-1/3}F\left(\frac{x}{y}, \frac{1}{y}\right).
\]
These solutions can also be obtained as integral periods via the integral representation (1.4),
\[
\eta_0(x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2} \, dt, \\
\eta_1(x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_{-\infty}^0 (-t)^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2} \, dt, \\
\eta_2(x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_{\frac{1}{\pi}}^{\frac{1}{\pi}} t^{a-1}(t-1)^{c-a-1}(xt-1)^{-b_1}(1-yt)^{-b_2} \, dt,
\]
for $a = b_1 = b_2 = \frac{1}{3}$, and $c = 1$.

**Remark 3.1.** Whenever the series defining the $\eta_i(x, y)$ do not converge, the value of $\eta_i(x, y)$ should be computed via the integral representations (3.3).

The class of identities we seek to establish has the form given in Definition 3.2.

**Definition 3.2.** The variables $m$, $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ are said to be related by a two-variable cubic modular equation of degree $\nu \in \mathbb{Z}[\omega]$ when the simultaneous relations (3.4)
\[
F(\beta_1, \beta_2)m = F(\alpha_1, \alpha_2) \\
F(1 - \beta_1, 1 - \beta_2)m = \nu \bar{\nu} F(1 - \alpha_1, 1 - \alpha_2) \\
\left(\beta_1^{-1/3}F\left(\frac{\beta_2}{\beta_1}, \frac{1}{\beta_1}\right) - \beta_2^{-1/3}F\left(\frac{\beta_1}{\beta_2}, \frac{1}{\beta_2}\right)\right) m = \nu \left(\alpha_1^{-1/3}F\left(\frac{\alpha_2}{\alpha_1}, \frac{1}{\alpha_1}\right) - \alpha_2^{-1/3}F\left(\frac{\alpha_1}{\alpha_2}, \frac{1}{\alpha_2}\right)\right)
\]
hold.

Note that we have five variables and three relations in these modular equations, so there are a total of two degrees of freedom in choosing the variables $m$, $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$. Note also that a one-variable cubic modular equation of degree $\nu \bar{\nu}$ may be obtained by specializing the two-variable modular equation.

**Proposition 3.3.** A one-variable modular equation of degree $\nu \bar{\nu}$ among $m$, $\alpha$ and $\beta$ may be obtained from a two-variable modular equation of degree $\nu$ among $m$, $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ by means of the substitutions
\[
\alpha_1 \rightarrow \alpha, \quad \alpha_2 \rightarrow \alpha, \\
\beta_1 \rightarrow \beta, \quad \beta_2 \rightarrow \beta, \\
m \rightarrow m.
\]

As indicated by the examples (1.1) and (1.2), the relationship between $m$, $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ that is induced by a two-variable cubic modular equation appears to be algebraic with the degree of the underlying algebraic relationship increasing with $|\nu|$. Theorem 3.4 makes the precise observation that when the defining relations of a two-variable cubic modular equation hold, $\mathbb{C}(m, \alpha_1, \alpha_2, \beta_1, \beta_2)$ is an algebraic extension of $\mathbb{C}(\alpha_1, \alpha_2)$ with degree bounded by $24(\nu \bar{\nu})^{18}$. This is, of course, an extremely pessimistic upper bound on the
degree, but is probably near the best one can do without considering the prime factors of \( \nu \) in \( \mathbb{Z}[\omega] \). The actual degree of this relationship is given by the index

\[
[\Gamma(\sqrt{-3}) : (D^{-1}\Gamma(\sqrt{-3})D) \cap \Gamma(\sqrt{-3})]
\]

where \( D \) is the diagonal matrix \( \text{diag}(1, \nu, \nu) \) and the group \( \Gamma(\sqrt{-3}) \), which is the monodromy group of functions \( \eta_1, \eta_1 \) and \( \eta_2 \), is defined in Definition 4.2.

**Theorem 3.4.** Suppose that the variables \( m, \alpha_1, \alpha_2, \beta_1, \) and \( \beta_2 \) are related by a two-variable cubic modular equation of degree \( \nu \in \mathbb{Z}[\omega] \). Set \( D \) to be the diagonal matrix \( \text{diag}(1, \nu, \nu) \) and set the group \( \Gamma(\sqrt{-3}) \) to be the congruence subgroup of the Picard modular group defined in Definition 4.2. We then have:

1. Any element \( X \in \mathbb{C}(m, \alpha_1, \alpha_2, \beta_1, \beta_2) \) is related to \( \alpha_1 \) and \( \alpha_2 \) by a polynomial equation \( f(X) = 0 \) where the coefficients of \( f(X) \) are in \( \mathbb{C}(\alpha_1, \alpha_2) \) and the degree of \( f(X) \) is exactly

   \[
d(\nu) := [\Gamma(\sqrt{-3}) : (D^{-1}\Gamma(\sqrt{-3})D) \cap \Gamma(\sqrt{-3})].
\]

   The polynomial \( f(X) \) may have repeated roots, and we have the bound \( d(\nu) \leq 24(\nu \nu)^{18} \).

2. The polynomials \( f(X) \) for the elements \( X = \beta_1 \) and \( X = \beta_2 \) do not have repeated roots in the cases \( \nu = \sqrt{-3} \) and \( \nu = 2 \). The degrees in these cases are \( d(\sqrt{-3}) = 9 \), and \( d(2) = 18 \).

3. Given that \( \beta_1 \) and \( \beta_2 \) are algebraically related to \( \alpha_1 \) and \( \alpha_2 \), the multiplier \( m \) is then algebraically related to \( \alpha_1 \) and \( \alpha_2 \) by the explicit formula

   \[
m^3 = (\nu^2 \nu) \left( \frac{\partial^3 \alpha_1 \partial \alpha_2}{\partial \beta_1 \partial \beta_2} - \frac{\partial^2 \alpha_1 \partial \alpha_2}{\partial \beta_2} \right) \frac{\beta_2^2}{(1 - \beta_1)^{2/3}} \frac{\beta_2^{2/3}}{(1 - \beta_2)^{2/3}} \frac{(\beta_1 - \beta_2)^{2/3}}{(\alpha_1 - \alpha_2)^{2/3}}.
\]

The third part of Theorem 3.4 should be compared to the analogous result for one-variable modular equations. If \( m, \alpha \) and \( \beta \) are related by a one-variable modular equation of degree \( n \) then the formula for the multiplier,

\[
m^2 = n \frac{d\alpha}{d\beta} \beta(1 - \beta) \alpha(1 - \alpha),
\]

follows easily as a corollary from Entry 30 in Chapter 11 of [1],

\[
\frac{d}{d\alpha} \left( \frac{2\pi}{\sqrt{3}} \frac{F(1 - \alpha)}{F(\alpha)} \right) = -\frac{1}{\alpha(1 - \alpha) F(\alpha)^2}.
\]

The first example of a two-variable modular equation is the three term iteration of [10] and [14]. If we start with three positive numbers \( a_0 \geq b_0 \geq c_0 \), and form three sequences according to ([14, p. 133])

\[
a_{n+1} = \left( a_n + b_n + c_n \right)/3,
\]

\[
b_{n+1}^3 + c_{n+1}^3 = a_{n+1}(a_nb_n + a_nc_n + b_nc_n) - a_nb_nc_n,
\]

then there is a common limit that satisfies

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = a_0 \left( 1 - \frac{b_0^3}{a_0^3}, 1 - \frac{c_0^3}{a_0^3} \right)^{-1}.
\]

The branches of the cube roots used to obtain \( b_{n+1} \) and \( c_{n+1} \) are not trivial, but they can be chosen so that even-indexed terms in the sequences are all real numbers. The fact that the common limit of the three term iteration can be expressed as the reciprocal of an \( F_1 \)
is equivalent to a modular equation for $F_1$, which can be stated as a modular equation of degree $\nu = \sqrt{-3}$.

**Theorem 3.5** (Proposition 2.5 of [14]). The following is a parameterization of the two-variable cubic modular equation of degree $1 + 2\omega = \sqrt{-3}$.

$$
\beta_1 = x^3, \quad \beta_2 = y^3,
$$

$$
\alpha_1 = 1 - \left(\frac{1 + \omega x + \omega y}{1 + x + y}\right)^3, \quad \alpha_2 = 1 - \left(\frac{1 + \omega x + \omega y}{1 + x + y}\right)^3,
$$

$$
m = 1 + x + y.
$$

We will also establish a modular equation of higher degree.

**Theorem 3.6.** The following is a parameterization of the two-variable cubic modular equation of degree 2.

$$
\beta_1 = \frac{x^3(y^2 + 3)(x^2 y^2 - 3x - 6y)}{(xy - 3)^3(xy + 3)}, \quad \beta_2 = \frac{y^3(x^2 + 3)(y x^2 - 3y - 6x)}{(xy - 3)^3(xy + 3)},
$$

$$
\alpha_1 = \frac{(x^2 + 3)(y^2 + 3)(x y^2 - 3y - 6x)}{(xy + 3)(xy - 3x - 3y - 3)^3}, \quad \alpha_2 = \frac{(y^2 + 3)(x^2 + 3)(x y^2 - 3x - 6y)}{(xy + 3)(xy - 3x - 3y - 3)^3},
$$

$$
m = \frac{xy - 3x - 3y - 3}{xy - 3}.
$$

The parameterizing variables can be given as

$$
x = 1 - m \left(\frac{1 - \alpha_1}{1 - \beta_1}\right)^{1/3}, \quad y = 1 - m \left(\frac{1 - \alpha_2}{1 - \beta_2}\right)^{1/3}.
$$

By Proposition 3.3, Theorems 3.5 and 3.6 reduce to Theorems 2.3 and 2.4 when $x = y$.

4. Picard Modular Forms and a Proof of Theorem 3.4

In order to prove that the modular equations in Definition 3.2 induce an algebraic relationship between $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$, it is necessary to recall some background facts concerning Picard modular functions. These functions are responsible for providing the automorphic functions that invert the Schwarz map mentioned in Section 3.

**Definition 4.1.** Set $\mathbb{B} = \{(\tau_1, \tau_2) \in \mathbb{C} P^2 : \tau_2 \bar{\tau}_2 < \tau_1 + \bar{\tau}_1\}$ and define the six $\Theta$ functions $\Theta_i : \mathbb{B} \to \mathbb{C}$ by

$$
\Theta_0(\tau_1, \tau_2) = \Theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Theta_3(\tau_1, \tau_2) = \Theta \begin{bmatrix} 0 & 0 & 0 \\ 2 & 5 & 6 \\ 2 & 6 & 5 \end{bmatrix},
$$

$$
\Theta_1(\tau_1, \tau_2) = \Theta \begin{bmatrix} 3 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Theta_4(\tau_1, \tau_2) = \Theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 6 & 5 \\ 1 & 5 & 6 \end{bmatrix},
$$

$$
\Theta_2(\tau_1, \tau_2) = \Theta \begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 0 \\ 6 & 5 & 5 \end{bmatrix}, \quad \Theta_5(\tau_1, \tau_2) = \Theta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},
$$

where

$$
\Theta \begin{bmatrix} a \\ b \end{bmatrix} = \sum_{n \in \mathbb{Z}^3} e^{\pi i(n \cdot a)(\Omega(n + a)^T + 2\pi i b(n + a)^T)}
$$

is a Riemann theta function of zero argument with characteristics $a, b \in \mathbb{R}^3$, and $\Omega$ is the $3 \times 3$ symmetric matrix with positive definite imaginary part

$$
\Omega(\tau_1, \tau_2) = \frac{1}{\omega - \bar{\omega}} \begin{pmatrix}
2\tau_1 - \tau_2^2 & (\omega - \bar{\omega}) \tau_2 & \tau_1 + \bar{\omega} \tau_2^2 \\
(\omega - \bar{\omega}) \tau_2 & 1 - \bar{\omega} & (\omega - 1) \tau_2 \\
\tau_1 + \bar{\omega} \tau_2^2 & (\omega - 1) \tau_2 & 2\tau_1 - \omega \tau_2^2
\end{pmatrix}.
$$
Definition 4.2. Set
\[ H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

The Picard modular group is defined as
\[ \Gamma = \{ g \in \text{GL}_3(\mathbb{Z}[\omega]) : g^T H g = H \}. \]

For any \( \nu \in \mathbb{Z}(\omega) \), set
\[ \Gamma(\nu) = \{ g \in \Gamma \mid g_{ij} \equiv \delta_{ij} \mod \nu \}. \]

Any matrix \( g = (g_{ij})_{ij} \in \Gamma \) acts on a point \( (\tau_1, \tau_2) \in \mathbb{B} \) via
\[ g : (\tau_1, \tau_2) \mapsto \left( \frac{g_{21} + g_{22} \tau_1 + g_{23} \tau_2}{g_{11} + g_{12} \tau_1 + g_{13} \tau_2}, \frac{g_{31} + g_{32} \tau_1 + g_{33} \tau_2}{g_{11} + g_{12} \tau_1 + g_{13} \tau_2} \right), \]
and the slash operator \( |g \) in weight \( k \) is defined as
\[ f|_g(\tau_1, \tau_2) = \frac{1}{(g_{11} + g_{12} \tau_1 + g_{13} \tau_2)^k} f(g(\tau_1, \tau_2)). \]

According to [8, p. 328–332], a list of generators for \( \Gamma(\sqrt{-3})/\{1, \omega, \omega^2\} \) can be given as \( \{g_1, g_2, g_3, g_4, g_5\} \), and if this list is augmented by \( \{g_6, g_7, g_8\} \), then it becomes a list of generators for \( \Gamma \), where
\[
\begin{align*}
g_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \\
g_2 &= \begin{pmatrix} 1 & \omega^2 - \omega & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
g_3 &= \begin{pmatrix} 1 & 0 & 0 \\ \omega^2 - \omega & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
g_4 &= \begin{pmatrix} 1 & -\omega & -\omega^2 \\ 0 & 1 & 0 \\ 0 & 1 & \omega \end{pmatrix}, \\
g_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 1 - \omega & 1 & \omega^2 - 1 \\ \omega - 1 & 0 & 1 \end{pmatrix}, \\
g_6 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
g_7 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
g_8 &= \begin{pmatrix} 1 & -\omega^2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

Lemma 4.3. The action of the generators \( g_i \) in weight one on each of the \( \Theta \) functions is summarized in the following table.

| \( f \) | \( f|_{g_1} \) | \( f|_{g_2} \) | \( f|_{g_3} \) | \( f|_{g_4} \) | \( f|_{g_5} \) | \( f|_{g_6} \) | \( f|_{g_7} \) | \( f|_{g_8} \) |
|---|---|---|---|---|---|---|---|---|
| \( \Theta_0 \) | \( \Theta_0 \) | \( \Theta_0 \) | \( \Theta_0 \) | \( \Theta_0 \) | \( \Theta_0 \) | \( \Theta_0 \) | \( \Theta_0 \) | \( \Theta_0 \) |
| \( \Theta_1 \) | \( \Theta_1 \) | \( \omega \Theta_1 \) | \( \Theta_1 \) | \( \omega \Theta_1 \) | \( \Theta_2 \) | \( \Theta_3 \) | \( \Theta_0 \) |
| \( \Theta_2 \) | \( \Theta_2 \) | \( \omega \Theta_2 \) | \( \Theta_2 \) | \( \Theta_2 \) | \( \Theta_1 \) | \( \Theta_4 \) | \( \Theta_1 \) |
| \( \Theta_3 \) | \( \Theta_3 \) | \( \omega \Theta_3 \) | \( \Theta_3 \) | \( \Theta_3 \) | \( \Theta_4 \) | \( \Theta_2 \) | \( -\omega \Theta_5 \) |
| \( \Theta_4 \) | \( \Theta_4 \) | \( \omega \Theta_4 \) | \( \Theta_4 \) | \( \Theta_4 \) | \( \Theta_3 \) | \( \Theta_2 \) | \( -\omega \Theta_5 \) |
| \( \Theta_5 \) | \( \omega \Theta_5 \) | \( \omega \Theta_5 \) | \( \omega^2 \Theta_5 \) | \( \omega^2 \Theta_5 \) | \( -\Theta_5 \) | \( -\Theta_5 \) | \( c^{-1} \omega \Theta_3 \) |

Here, \( c \) is a constant with absolute value 1.

Lemma 4.4 ( p. 349 of [8] ). For the group \( \Gamma(\sqrt{-3}) \), we have:

1. The graded ring of holomorphic modular forms of weight \( \{3k\}_{k=0}^\infty \) with respect to \( \Gamma(\sqrt{-3}) \) is given by \( \mathbb{C}[\xi_0, \xi_1, \xi_2, \Delta] / (\Delta^3 - \xi_0 \xi_1 \xi_2 (\xi_1 - \xi_0)(\xi_0 - \xi_2)(\xi_2 - \xi_1)) \), where
   \[
   \begin{align*}
   \xi_0(\tau_1, \tau_2) &= \Theta_0^3, \\
   \xi_1(\tau_1, \tau_2) &= \Theta_1^3, \\
   \xi_2(\tau_1, \tau_2) &= \Theta_2^3, \\
   \Delta(\tau_1, \tau_2) &= \omega^{1/3} \Theta_0 \Theta_1 \Theta_2 \Theta_3 \Theta_4 \Theta_5.
   \end{align*}
   \]
Lemma 4.5. If \( q = e^{-\frac{2\pi}{\sqrt{3}}}, \) then the series expansions of the theta functions are as follows.

\[
\begin{align*}
\Theta_0(\tau_1, \tau_2) &= \sum_{\mu \in \mathbb{Z}[\omega]} \sum_{z \in \mathbb{Z}+1/6} q^{\mu z} e^{\frac{\pi i}{\sqrt{3}} \tau_1^2 \mu^2 + 2\pi i z (\tau_1 + \tau_2) + \pi i z^2}, \\
\Theta_1(\tau_1, \tau_2) &= \sum_{\mu \in \mathbb{Z}[\omega]} \sum_{z \in \mathbb{Z}+1/6} q^{\mu z} e^{\frac{\pi i}{\sqrt{3}} \tau_1^2 \mu^2 + 2\pi i z (\tau_1 + \tau_2) + \pi i z^2}, \\
\Theta_2(\tau_1, \tau_2) &= \sum_{\mu \in \mathbb{Z}[\omega]} \sum_{z \in \mathbb{Z}+1/6} q^{\mu z} e^{\frac{\pi i}{\sqrt{3}} \tau_1^2 \mu^2 + 2\pi i z (\tau_1 + \tau_2) + \pi i z^2}, \\
\Theta_3(\tau_1, \tau_2) &= \sum_{\mu \in \mathbb{Z}[\omega]} \sum_{z \in \mathbb{Z}+1/6} q^{\mu z} e^{\frac{\pi i}{\sqrt{3}} \tau_1^2 \mu^2 + 2\pi i z (\tau_1 + \tau_2) + \pi i z^2}, \\
\Theta_4(\tau_1, \tau_2) &= \sum_{\mu \in \mathbb{Z}[\omega]} \sum_{z \in \mathbb{Z}+1/6} q^{\mu z} e^{\frac{\pi i}{\sqrt{3}} \tau_1^2 \mu^2 + 2\pi i z (\tau_1 + \tau_2) + \pi i z^2}, \\
\Theta_5(\tau_1, \tau_2) &= \sum_{\mu \in \mathbb{Z}[\omega]} \sum_{z \in \mathbb{Z}+1/6} q^{\mu z} e^{\frac{\pi i}{\sqrt{3}} \tau_1^2 \mu^2 + 2\pi i z (\tau_1 + \tau_2) + \pi i z^2}.
\end{align*}
\]

Proof. These follow from the definition

\[
\Theta \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} (\Omega(\tau_1, \tau_2)) = \sum_{(x,y,z) \in \mathbb{Z}^3 + (a_1, a_2, a_3)} \exp \left( \pi i (x, y, z) \cdot \Omega(\tau_1, \tau_2) \cdot (x, y, z)^T + 2\pi i (x, y) \cdot (b_1, b_2, b_3)^T \right),
\]

and by setting \( \mu = x - \omega^2 y. \)

Note that these functions reduce to the Borweins’ cubic theta functions when \( \tau_2 = 0. \) More precisely, when \( \tau_2 = 0, \) they are a constant multiple of \( a(q), c(q), c(q), b(q), b(q) \) and 0, respectively.

Lemma 4.6. We have the following identities.

\[
\begin{align*}
\Theta_3(\tau_1, \tau_2)^3 &= \Theta_0(\tau_1, \tau_2)^3 - \Theta_1(\tau_1, \tau_2)^3, \\
\Theta_4(\tau_1, \tau_2)^3 &= \Theta_0(\tau_1, \tau_2)^3 - \Theta_2(\tau_1, \tau_2)^3, \\
\omega \Theta_5(\tau_1, \tau_2)^3 &= \Theta_1(\tau_1, \tau_2)^3 - \Theta_2(\tau_1, \tau_2)^3, \\
\Theta_0(\tau_1, \tau_2) &= C \prod_{\Theta_i(\tau_1, \tau_2)^3} \Theta_0(\tau_1, \tau_2)^3)
\end{align*}
\]

for \( \Theta_0 \geq \Theta_1 \geq \Theta_2 \geq 0, \)

where \( C = \cdot \).

Proof. The first three follow from Lemma 4.4. Each of the functions \( \Theta_0^3, \Theta_1^3, \) and \( \Theta_2^3 \) is a modular form of weight 3 with respect to \( \Gamma(\sqrt{-3}). \) They must be linear combinations of \( \Theta_0, \Theta_1, \) and \( \Theta_2 \) since these three functions span this space. The coefficients of these linear combinations may be found with the series expansions in Lemma 4.5. The last identity...
is equivalent to Corollary 2.1 in [10], and it must be considered a formal identity because \( \eta_0(\lambda_1, \lambda_2) \) is in general a multi-valued function. On the branch fixed in (3.2), the equality

\[
\Theta_0 = C F \left( \frac{\Theta_1^4}{\Theta_2^3}, \frac{\Theta_3^3}{\Theta_0^3} \right)
\]

holds at least when \( \Theta_0 \geq \Theta_1 \geq \Theta_2 \geq 0 \).

**Proposition 4.7** (Catalog of \( \Theta \) function evaluations). Suppose that \( m, \alpha, \alpha_1 \beta_1 \) and \( \beta_2 \) are related by a two-variable cubic modular equation of degree \( \nu \). We then have the following table for converting identities among \( \Theta \) functions to modular equations and vice-versa.

\[
\begin{align*}
\Theta_0(\tau_1, \tau_2) &= z, & \Theta_0(\nu \bar{\tau}_1, \nu \bar{\tau}_2) &= m^{-1} z \\
\Theta_1(\tau_1, \tau_2) &= \alpha_1^{1/3} z, & \Theta_1(\nu \bar{\tau}_1, \nu \bar{\tau}_2) &= m^{-1/3} \beta_1 z \\
\Theta_2(\tau_1, \tau_2) &= \alpha_2^{1/3} z, & \Theta_2(\nu \bar{\tau}_1, \nu \bar{\tau}_2) &= m^{-1/3} \beta_2 z \\
\Theta_3(\tau_1, \tau_2) &= (1 - \alpha_1)^{1/3} z, & \Theta_3(\nu \bar{\tau}_1, \nu \bar{\tau}_2) &= m^{-1}(1 - \beta_1)^{1/3} z \\
\Theta_4(\tau_1, \tau_2) &= (1 - \alpha_2)^{1/3} z, & \Theta_4(\nu \bar{\tau}_1, \nu \bar{\tau}_2) &= m^{-1}(1 - \beta_2)^{1/3} z \\
\Theta_5(\tau_1, \tau_2) &= (\alpha_1 - \alpha_2)^{1/3} \omega^{-1/3} z, & \Theta_5(\nu \bar{\tau}_1, \nu \bar{\tau}_2) &= m^{-1}(\beta_1 - \beta_2)^{1/3} \omega^{-1/3} z,
\end{align*}
\]

where

\[
z = C F(\alpha_1, \alpha_2)
\]

**Proof.** The fundamental inversion formula given in [7, p. 131] and [8, p. 327] states that

\[
\alpha_1 = \frac{\Theta_1(\tau_1, \tau_2)^3}{\Theta_0(\tau_1, \tau_2)^3}, \quad \alpha_2 = \frac{\Theta_2(\tau_1, \tau_2)^3}{\Theta_0(\tau_1, \tau_2)^3}
\]

when

\[
\tau_1 = \frac{F(1 - \alpha_1, 1 - \alpha_2)}{F(\alpha_1, \alpha_2)},
\]

\[
\tau_2 = \frac{\alpha_1^{-1/3} F(\alpha_2, 1 - \alpha_1) - \alpha_2^{-1/3} F(1, \alpha_1 \alpha_2)}{F(\alpha_1, \alpha_2)}.
\]

All of the conversions in this table follow from this inversion formula, the definition of the two-variable modular equation, and the identities given in Lemma 4.6.

**Lemma 4.8.** Suppose that \( f(\tau_1, \tau_2) \) is a (weight zero) modular function with respect to \( \Gamma \). Then for any \( \nu \in \mathbb{Z}(\omega) \), \( f(\nu \bar{\tau}_1, \nu \bar{\tau}_2) \) is a modular function with respect to \( \Gamma(\nu \bar{\nu}) \).

**Proof.** For any \((g_{ij})_{ij} \in \Gamma(\nu \bar{\nu})\) we need to verify that

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \nu \bar{\nu} & 0 \\
0 & 0 & \nu
\end{pmatrix}
\begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \nu \bar{\nu} & 0 \\
0 & 0 & \nu
\end{pmatrix}^{-1}
\in \Gamma.
\]

This matrix is

\[
\begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix}
\begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix}
\begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix}
\]

which is clearly in \( \Gamma \) if \( g_{ij} \equiv \delta_{ij} \mod \nu \bar{\nu} \).

**Lemma 4.9.** For any \( \nu \in \mathbb{Z}(\omega) \), \( \Gamma(\nu) \) is a normal subgroup of \( \Gamma \) with \(|\Gamma : \Gamma(\nu)| \leq (\nu \bar{\nu})^9 \).
Proof. The group $\Gamma(\alpha)$ is the kernel of the map $\phi : \Gamma \to PGL_3(\mathbb{Z}[\omega]/(\nu))$, which is obtained by reducing elements point-wise as

$$\phi : (g_{ij}) \mapsto (g_{ij} \mod \nu)_{ij}.$$ 

Thus, $|\Gamma/\Gamma(\nu)| \leq |PGL_3(\mathbb{Z}[\omega]/(\nu))| \leq (\nu\bar{\nu})^9$.

**Proposition 4.10.** The first and second parts of Theorem 3.4 hold.

Proof. By Proposition 4.7, the variables in a two-variable cubic modular equation are parameterized by $(\tau_1, \tau_2) \in \mathbb{B}$ as

$$\alpha_1 = \frac{\Theta_1(\tau_1, \tau_2)}{\Theta_0(\tau_1, \tau_2)^3}, \quad \alpha_2 = \frac{\Theta_2(\tau_1, \tau_2)}{\Theta_0(\tau_1, \tau_2)^3},$$

$$\beta_1 = \frac{\Theta_1(\nu\bar{\nu}\tau_1, \nu\tau_2)}{\Theta_0(\nu\bar{\nu}\tau_1, \nu\tau_2)^3}, \quad \beta_2 = \frac{\Theta_2(\nu\bar{\nu}\tau_1, \nu\tau_2)}{\Theta_0(\nu\bar{\nu}\tau_1, \nu\tau_2)^3},$$

$$m = \frac{\Theta_0(\tau_1, \tau_2)}{\Theta_0(\nu\bar{\nu}\tau_1, \alpha\tau_2)}.$$

Recall that $D$ was the diagonal matrix $\text{diag}(1, \nu\bar{\nu}, \nu)$. An element $X \in \mathbb{C}(m, \alpha_1, \alpha_2, \beta_2, \beta_2)$ is thus pushed forward to a modular function $X(\tau_1, \tau_2)$ with respect to the group

$$(D^{-1}\Gamma(\sqrt{-3}D)) \cap \Gamma(\sqrt{-3}).$$

Let us first obtain the bound on the index $d(\nu)$. We have,

$$d(\nu) = [\Gamma(\sqrt{-3}) : (D^{-1}\Gamma(\sqrt{-3}D)) \cap \Gamma(\sqrt{-3})] \leq [\Gamma : \Gamma(\sqrt{-3})][\Gamma : (D^{-1}\Gamma D) \cap \Gamma]

\leq [\Gamma : \Gamma(\sqrt{-3})][\Gamma : \Gamma(\nu\bar{\nu})] \leq 24(\nu\bar{\nu})^{18}.$$ 

Now let $\Gamma(\sqrt{-3}) = \bigcup_i ((D^{-1}\Gamma(\sqrt{-3}D)) \cap \Gamma(\sqrt{-3}))M_i$ be a decomposition of $\Gamma(\sqrt{-3})$ into right cosets with $M_0 = I$. The polynomial $f(x)$ is then

$$f(x) = \prod_i (x - X|_{DM_i(\tau_1, \tau_2)}).$$

By Lemma 4.4, the coefficients of $f(x)$ are rational functions of $\alpha_1$ and $\alpha_2$. From the factor with $i = 0$, we see that $f(x)$ has $x = X(\tau_1, \tau_2)$ as a root.

The second part of Theorem 3.4 follows from a straightforward calculation. The conjugates under the action of $\Gamma(\sqrt{-3})$ of

$$\beta_1 = \frac{\Theta_1(\nu\bar{\nu}\tau_1, \nu\tau_2)}{\Theta_0(\nu\bar{\nu}\tau_1, \nu\tau_2)^3}, \quad \beta_2 = \frac{\Theta_2(\nu\bar{\nu}\tau_1, \nu\tau_2)}{\Theta_0(\nu\bar{\nu}\tau_1, \nu\tau_2)^3},$$

which are $d(\nu)$ in number, are all distinct in the cases $\nu = \sqrt{-3}$ and $\nu = 2$.

5. Partial Differential Equations and the Algebracity of the Multiplier

In [14], the fundamental identity needed to establish the $F_1$ as the common limit of the three term iteration, namely

$$F(1 - x^3, 1 - y^3) = \frac{3}{1 + x + y}F\left(\frac{1 + \omega x + \omega^2 y}{1 + x + y}, \frac{1 + \omega^2 x + \omega y}{1 + x + y}\right)^3$$

was establish by a direct appeal to the system of partial differential equations (3.1). The approach we will take to prove the degree 2 modular equation in Theorem 3.6 will use the four derivatives associated to $GL(3)$ introduced in [16]. In doing so we will find that when
Proof. The “if” part of the proposition is clear from the calculations in (5.3). To prove the “only if” part, consider the function

\[ f(x, y) = \frac{\eta_0(x, y)}{W_{x,y}(\eta)^{1/3}}, \]

Proposition 5.1. We have that \( W_{x,y}(\eta) = W_{(x,y)}(\hat{\eta}) \) if and only if \( \hat{\eta} = m(x,y)M \cdot \eta \) for some scalar function \( m(x,y) \) and a constant 3 \times 3 matrix \( M \).

Proof. The “if” part of the proposition is clear from the calculations in (5.3). To prove the “only if” part, consider the function
and set 
\[(a, b, 3c, 3d) = W_{(x,y)}(\eta) = W_{(x,y)}(\hat{\eta}).\]

Then the function \(f(x, y)\) satisfies the differential equations (see [16])
\[
\begin{align*}
f_{xx} &= (2c^2 - 2ad - ay + cx) f - cx f_x + af_y, \\
f_{xy} &= (cd - ab - cy + dx) f - dx f_x + cf_y, \\
f_{yy} &= (2d^2 - 2bc - dy + bx) f - bx f_x + df_y.
\end{align*}
\]

Since the coefficients of this differential equation only depend on \(a, b, c,\) and \(d\), it is also satisfied by any of the functions
\[
f(x, y) = \frac{\eta_i(x, y)}{W_{x,y}(\eta)^{1/3}} \quad \text{or} \quad f(x, y) = \frac{\hat{\eta}_i(x, y)}{W_{x,y}(\hat{\eta})^{1/3}}
\]
for \(i = 0, 1, 2.\) However, since there are only three linearly independent solutions, there must be a constant matrix \(M\) such that
\[
\frac{\hat{\eta}(x, y)}{W_{x,y}(\hat{\eta})^{1/3}} = M \cdot \frac{\eta(x, y)}{W_{x,y}(\eta)^{1/3}}.
\]

Thus, the “only if” implication is proven with
\[
m(x, y) = \frac{W_{x,y}(\eta)^{1/3}}{W_{x,y}(\hat{\eta})^{1/3}}.
\]

\[\square\]

6. Proofs of the modular equations via the hypergeometric differential equation

Proposition 6.1 provides a connection formula for the Wronskians and is necessary in order to complete the calculations for a given modular equation.

**Proposition 6.1.** We have the following formulas for the change of variables in the Wronskians:
\[
W_{x_1,x_2}(\eta) = \left(\frac{\partial y_1}{\partial x_1} \frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2} \frac{\partial y_2}{\partial x_1}\right) W_{y_1,y_2}(\eta),
\]
\[
W_{x_1,x_2}(\tilde{\eta}) = T_{x_1,x_2}^{y_1,y_2} \cdot W_{y_1,y_2}(\eta) + W_{x_1,x_2}((1, y_1, y_2)),
\]
where
\[
T_{x_1,x_2}^{y_1,y_2} := \left(\frac{\partial y_1}{\partial x_1} \frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2} \frac{\partial y_2}{\partial x_1}\right)^{-1} \times
\[
\begin{pmatrix}
\frac{\partial y_1}{\partial x_1}\frac{\partial^3 y_1}{\partial x_1^3} & \frac{\partial y_1}{\partial x_1}\frac{\partial^3 y_2}{\partial x_1^3} & \frac{\partial y_1}{\partial x_1}\frac{\partial^2 y_2}{\partial x_1^2} & \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_1} \\
\frac{\partial y_2}{\partial x_2}\frac{\partial^3 y_1}{\partial x_2^3} & \frac{\partial y_2}{\partial x_2}\frac{\partial^3 y_2}{\partial x_2^3} & \frac{\partial y_2}{\partial x_2}\frac{\partial^2 y_2}{\partial x_2^2} & \frac{\partial y_2}{\partial x_2}\frac{\partial y_2}{\partial x_2} \\
\frac{3}{2}\frac{\partial y_1}{\partial x_1}\frac{\partial^2 y_1}{\partial x_1^2} & \frac{3}{2}\frac{\partial y_1}{\partial x_1}\frac{\partial^2 y_2}{\partial x_1^2} & \frac{3}{2}\frac{\partial y_1}{\partial x_1}\frac{\partial^2 y_2}{\partial x_1^2} & \frac{3}{2}\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_1} \\
\frac{3}{2}\frac{\partial y_2}{\partial x_2}\frac{\partial^2 y_1}{\partial x_2^2} & \frac{3}{2}\frac{\partial y_2}{\partial x_2}\frac{\partial^2 y_2}{\partial x_2^2} & \frac{3}{2}\frac{\partial y_2}{\partial x_2}\frac{\partial^2 y_2}{\partial x_2^2} & \frac{3}{2}\frac{\partial y_2}{\partial x_2}\frac{\partial y_2}{\partial x_2}
\end{pmatrix}.
\]

**Proof.** The components of \(W_{x_1,x_2}(\eta)\) are \(-a_2, -c_1, -a_1 + 2b_2,\) and \(c_2 - 2b_1\) where \(\eta(x_1, x_2)\) form a basis for the solutions of the system
\[
\eta_{x_1} = a_0 \eta + a_1 \eta_{x_1} + a_2 \eta_{x_2},
\]
\[
\eta_{x_2} = b_0 \eta + b_1 \eta_{x_1} + b_2 \eta_{x_2},
\]
\[
\eta_{x_2} = c_0 \eta + c_1 \eta_{x_1} + c_2 \eta_{x_2}.
\]
By changing the variables in the system to $y_1$ and $y_2$, we obtain a new system of the form

$$
\begin{align*}
\eta_{y_1 y_1} &= A_0 \eta + A_1 \eta y_1 + A_2 \eta y_2, \\
\eta_{y_1 y_2} &= B_0 \eta + B_1 \eta y_1 + B_2 \eta y_2, \\
\eta_{y_2 y_2} &= C_0 \eta + C_1 \eta y_1 + C_2 \eta y_2.
\end{align*}
$$

Calculations show that the components of $W_{y_1 y_2}(\eta)$, which are $A_2$, $-C_1$, $-A_1 + 2B_2$, and $C_2 - 2B_1$, are related to the components of $W_{x_1 x_2}(\eta)$ in the stated manner when the dependent variables are changed from $(x_1, x_2)$ to $(y_1, y_2)$. 

We now set $\eta(x_1, x_2)$ to a basis of solutions to system in (3.1), i.e.

$$
\begin{align*}
\eta_0(x_1, x_2) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 t^{a-1}(1 - t)^{c-a-1}(1 - x_1 t)^{-b_1}(1 - x_2 t)^{-b_2} \, dt, \\
\eta_1(x_1, x_2) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_{-\infty}^0 (-t)^{a-1}(1 - t)^{c-a-1}(1 - x_1 t)^{-b_1}(1 - x_2 t)^{-b_2} \, dt, \\
\eta_2(x_1, x_2) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_{\frac{1}{x_2}}^1 t^{a-1}(1 - t)^{c-a-1}(x_1 t - 1)^{-b_1}(1 - x_2 t)^{-b_2} \, dt.
\end{align*}
$$

The components of $W_{x_1 x_2}(\eta)$ may be read directly from the coefficients on the right hand side of (3.1). Thus,

$$
W_{x_1 x_2}(\eta) = \left( \begin{array}{c}
\frac{x^3(y^2 + 3)(y^2 - 3x - 6y)}{(xy - 3)^3(xy + 3)}, \\
\frac{y^3(x^2 + 3)(xy^2 - 3y - 6x)}{(xy - 3)^3(xy + 3)}
\end{array} \right).
$$

In order to prove the two-variable cubic modular equation in Theorem 3.6, we set $a = b_1 = b_2 = 1/3$ and $c = 1$ and, via Proposition 6.1, we must evaluate

$$
W_{x,y}(\eta(x^3(y^2 + 3)(y^2 - 3x - 6y), y^3(x^2 + 3)(xy^2 - 3y - 6x))
$$

and

$$
W_{x,y}(\eta(x^3(y^2 + 3)(y^2 - 3x - 6y), y^3(x^2 + 3)(xy^2 - 3y - 6x))
$$

Fortunately the four rational functions obtained in each case do agree, and so we have a valid modular equation up to an evaluation of the constant matrix $M$ in Proposition 5.1. Let us first obtain the proposed multiplier $m$ via (5.1), which is a simple consequence of the evaluation of $W_{x_1 x_2}(\eta(x_1, x_2))$, which has been given in [15]. The result is that

$$
W_{x_1 x_2}(\eta) = \frac{C(a, b_1, b_2, c)}{x_1^{c-b_2}x_2^{c-b_1}(1 - x_1)^{a+b_1-c+1}(1 - x_2)^{a+b_2-c+1}(x_1 - x_2)^{b_1+b_2}},
$$

where $C(1/3, 1/3, 1/3, 1)$ is a non-zero constant. Thus, by Proposition 5.1,

$$
\det \begin{pmatrix} m^{-1} & 0 & 0 \\ 0 & m^{-1}v & 0 \\ 0 & 0 & m^{-1}\nu \end{pmatrix} = \frac{W_{\alpha_1, \alpha_1}(\eta(\beta_1, \beta_2))}{W_{\alpha_1, \alpha_1}(\eta(\alpha_1, \alpha_2))} = \frac{\partial_1 \beta_1 \partial_2 \beta_2}{\partial_1 \beta_1 \partial_2 \alpha_2 - \partial_2 \beta_2 \partial_1 \alpha_1} \frac{\alpha_1^{2/3}(1 - \alpha_1)^{2/3} \alpha_2^{2/3}(1 - \alpha_2)^{2/3}}{\beta_1^{2/3}(1 - \beta_1)^{2/3} \beta_2^{2/3}(1 - \beta_2)^{2/3}} \frac{(\alpha_1 - \alpha_2)^{2/3}}{(\beta_1 - \beta_2)^{2/3}},
$$

and so (5.1) is clear.
Let \( m, \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) be the functions of \( x \) and \( y \) in Theorem 3.6. In order to complete the proof of this modular equation, we need to show that
\[
\begin{pmatrix}
\eta_0(\beta_1, \beta_2) \\
\eta_1(\beta_1, \beta_2) \\
\eta_2(\beta_1, \beta_2)
\end{pmatrix} = M \cdot \begin{pmatrix}
1m^{-1} \eta_0(\alpha_1, \alpha_2) \\
4m^{-1} \eta_1(\alpha_1, \alpha_2) \\
2m^{-1} \eta_2(\alpha_1, \alpha_2)
\end{pmatrix}
\]
implies that \( M \) is the identity matrix. This can be accomplished by setting \( x = x(t) \) and \( y = y(t) \) and comparing the asymptotic expansions for small \( t \). If \( x = t \) and \( y = t \), then the vectors on the left and right are both approximately \((1, c \log t, 0)\) where \( c \) is a nonzero constant, possibly different for each side. Thus, \( M_{11} = 1, \) and \( M_{12} = 0. \) Next, if \( x = t \) and \( y = 0 \), then both vectors are approximately \((1, c \log t, 1)\). Thus, \( M_{13} = M_{32} = 0. \) Next, if \( x = 1 + t \) and \( y = 1 \), then both vectors are approximately \((c \log t, 1, 1/4)\). Thus, \( M_{21} = M_{31} = 0, M_{33} = 1, \) and \( 1 = M_{22} + M_{23}/4. \) Finally, if \( x = 1 - t \) and \( y = 1 - t \), then both vectors are approximately \((c \log t, 1, 0)\). Thus, \( M_{22} = 1, \) and by the previous relation, \( M_{23} = 0. \)

7. PROOFS OF THE TWO-VARIABLE MODULAR EQUATIONS VIA THETA FUNCTIONS

Although usage of the hypergeometric differential equation in Section 6 is capable of producing valid proofs of Theorems 3.5 and 3.6, it would be much more desirable to have motivated proofs of these modular equations. The goal of this section is to state and prove identities of \( \Theta \) functions and then transfer them into modular equations via the catalog of evaluations in Proposition 4.7.

**Theorem 7.1.** The following \( \Theta \) function identities of degree \( 1 + 2\omega = \sqrt{-3} \) hold.
\[
\begin{align*}
3\Theta_0(3\tau_1, (1 + 2\omega)\tau_2) &= \Theta_0(\tau_1, \tau_2) + \Theta_3(\tau_1, \tau_2) + \Theta_4(\tau_1, \tau_2), \\
3\Theta_1(3\tau_1, (1 + 2\omega)\tau_2) &= \Theta_0(\tau_1, \tau_2) + \Theta_3(\tau_1, \tau_2) + \bar{\omega}\Theta_4(\tau_1, \tau_2), \\
3\Theta_2(3\tau_1, (1 + 2\omega)\tau_2) &= \Theta_0(\tau_1, \tau_2) + \bar{\omega}\Theta_3(\tau_1, \tau_2) + \omega\Theta_4(\tau_1, \tau_2).
\end{align*}
\]

*Proof. Should be easy.*

**Theorem 7.2.** The following \( \Theta \) function identities of degree \( 2 \) hold.
\[
\begin{align*}
3\Theta_0(4\tau_1, 2\tau_2)\Theta_5(4\tau_1, 2\tau_2) &= (\Theta_0(\tau_1, \tau_2) - \Theta_0(4\tau_1, 2\tau_2))(\Theta_5(4\tau_1, 2\tau_2) + \Theta_5(\tau_1, \tau_2)), \\
3\Theta_1(4\tau_1, 2\tau_2)\Theta_4(4\tau_1, 2\tau_2) &= (\Theta_3(\tau_1, \tau_2) - \Theta_4(4\tau_1, 2\tau_2))(\Theta_1(4\tau_1, 2\tau_2) - \Theta_2(\tau_1, \tau_2)), \\
3\Theta_2(4\tau_1, 2\tau_2)\Theta_3(4\tau_1, 2\tau_2) &= (\Theta_1(\tau_1, \tau_2) - \Theta_2(4\tau_1, 2\tau_2))(\Theta_3(4\tau_1, 2\tau_2) - \Theta_4(\tau_1, \tau_2)).
\end{align*}
\]

*Proof. Should not be easy.*

*Proof of Theorem 3.5.* Applying Proposition 4.7 to the three equalities in Theorem 7.1 gives
\[
\begin{align*}
3m^{-1} &= 1 + (1 - \alpha_1)^{1/3} + (1 - \alpha_2)^{1/3}, \\
3m^{-1}\beta_1^{1/3} &= 1 + \omega(1 - \alpha_1)^{1/3} + \bar{\omega}(1 - \alpha_2)^{1/3}, \\
3m^{-1}\beta_2^{1/3} &= 1 + \bar{\omega}(1 - \alpha_1)^{1/3} + \omega(1 - \alpha_2)^{1/3}.
\end{align*}
\]
The parameterizations in Theorem 3.5 are easily seen to be equivalent to these three relations.

*Proof of Theorem 3.6.* Let us introduce parameters \( x \) and \( y \) defined by
\[
\begin{align*}
m \left( \frac{1 - \alpha_1}{1 - \beta_2} \right)^{1/3} &= 1 - x, \\
(7.1) \\
m \left( \frac{1 - \alpha_2}{1 - \beta_1} \right)^{1/3} &= 1 - y
\end{align*}
\]
The last two equations in Theorem 7.2, when transferred using Proposition 4.7 and the definitions of $x$ and $y$, become

$$m \left( \frac{\alpha_2}{\beta_1} \right)^{1/3} = 1 + \frac{3}{x},$$

(7.2)

$$m \left( \frac{\alpha_1}{\beta_2} \right)^{1/3} = 1 + \frac{3}{y}.$$  

The first equation in Theorem 7.2 directly becomes

$$m \left( \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} \right)^{1/3} = \frac{3}{m - 1}.$$  

(7.3)

We first solve (7.2) for $\beta_1$ and $\beta_2$ in terms of $m$, $x$, $y$, $\alpha_1$ and $\alpha_2$. We then substitute this solution for $(\beta_1, \beta_2)$ into (7.1) and solve the resulting linear equations for $(\alpha_1, \alpha_2)$ in terms of $m$, $x$, and $y$ alone. The resulting solutions for $(\alpha_1, \alpha_2)$ (and thus $(\beta_1, \beta_2)$) are

$$\alpha_1 = \frac{(y + 3)^3 (m^3 + (x - 1)^3)}{m^3 (x (x^2 - 3x + 3) y^3 + 9y^2 + 27y + 27)},$$

$$\alpha_2 = \frac{(x + 3)^3 (m^3 + (y - 1)^3)}{m^3 (x^2y (y^2 - 3y + 3) + 9x^2 + 27x + 27)},$$

$$\beta_1 = \frac{x^3 (m^3 + (y - 1)^3)}{x^3y (y^2 - 3y + 3) + 9x^2 + 27x + 27},$$

$$\beta_2 = \frac{y^3 (m^3 + (x - 1)^3)}{x (x^2 - 3x + 3) y^3 + 9y^2 + 27y + 27}.$$

When these solutions are substituted into (7.3), we can solve for $m$ in terms of $x$ and $y$ alone, and then obtain $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ in terms of $x$ and $y$ alone by substituting the obtained solution for $m$. The result is the parameterizations in Theorem 3.6. □

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