Continuous Distributions Arising from the Three Gaps Theorem

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Abstract

The well known Three Gap Theorem states that there are at most three gap sizes in the sequence of fractional parts \( \{\alpha n\}_{n<N} \). The main discovery in this paper is that if we average over a short interval \([\alpha - \delta, \alpha + \delta]\), the distribution becomes continuous. Moreover, this continuous distribution is universal in the sense that it is the same for any \( \alpha \) and any interval around \( \alpha \). Under these circumstances one would expect that the above averaging process would introduce enough randomness in the sequence so that the limiting distribution would be Poissonian. We will prove that, surprisingly, this is not the case.

1 Introduction

Let \( \alpha \) be an irrational number. Consider the fractional parts \( \{n\alpha\}_{0 \leq n < Q} \) arranged in increasing order and placed on the interval \([0, 1]\) with 0 and 1 identified so that \( Q \) gaps appear. If we denote this sequence by \( S_Q(\alpha) \) and consider the gaps between consecutive elements, then it is well known (see van Ravenstein [23], Sós [28] and Świerczkowski [29]) that there are at most three gap sizes in \( S_Q(\alpha) \). The sequence \( \{n^d\alpha\} \), for \( d \geq 1 \), has been extensively studied. If \( d > 1 \), the main reference is the classical work by Rudnick and Sarnak [24], in which the authors proved that for almost all \( \alpha \) the pair correlation of members of the sequence is poissonian (see also [5]). However, the sequence \( \{n\alpha\} \) does not give rise to a poissonian distribution.

Prior and after the work of Rudnick and Sarnak [24], various other works have been done studying fractional parts of polynomials. A non-exhaustive list of references includes Arhipov, Karacuba, and Čubarikov [2], Baker and Harman [3], de Velasco [7], Karacuba [16], Kovalevskaja [17], Moshchevitin [20], Schmidt [27], and Wooley [31]. The distribution of gaps and other aspects of fractional parts has been also investigated for specific types of numbers including real algebraic numbers (Misevičius and Vakrinienė [19]), Salem numbers (Zaimi [32]), rational powers (Dubickas [8]), perfect square multiples of \( \alpha \), i.e. \( \alpha n^2 \) (Heath-Brown [14] and Truelsen [30]), Pisot numbers (Dubickas [9]) and others. The case \( d = 1 \), our sequence of interest \( \{n\alpha\} \), has been the subject of extensive studies both in the past and in recent years (See Pillishshammer [22] and the references there in).

In this paper we are interest in short averages of the nearest neighbor distribution of elements in \( S_Q(\alpha) \). We denote the \( n^{\text{th}} \) element of \( S_Q(\alpha) \) by \( \{\sigma_n, \alpha\} \), one consider function:

\[
g_{[a,b]}(\lambda; Q) := \frac{1}{b-a} \int_a^b \frac{\# \{0 \leq n < Q : \{\sigma_{n+1} \alpha\} - \{\sigma_n \alpha\} \geq \lambda / Q \}}{Q} d\alpha. \tag{1.1}
\]
Is it a continuous function of $\lambda$? Is it differentiable? When $Q$ is taken to be 1000, $\alpha = 1/3$, and $\delta = 1/10$, the resulting graph is shown in Figure 1. It appears that the limit does exist and is a continuous function of $\lambda$. Moreover, the graphs for different values of $\alpha$ and $\delta$ appear to be identical.

This distribution evaded some of the methods that have proven to be efficient for other cases. Heuristically speaking, it is reasonable to think that the subtleties lie on the nature of the sequence and how the gaps are related to its elements. One would suspect that connecting the elements of the sequence to the three possible gaps would produce a successful approach to solve this problem. Indeed, this is what we do. We establish such a connection using the classical properties of Farey Series that can be found in Hall [10], Hardy and Wright [13] and LeVeque [18]. We also use other further developed properties connecting Farey Series with Kloosterman sums which have been applied to some useful asymptotics in [1], Hall [11] and in Hall and Tenenbaum [12]. With these tools in hand we find a striking formula for the $g_1$ distribution function and show that this function is still approached as long as the size of the interval goes to zero no faster than $Q^{-1/2}$. More precisely we prove the following:

**Theorem 1.1.** As $Q \to \infty$, the nearest neighbor distribution in (1.1) is independent of $[a,b]$, and

![Figure 1: $g_1^{[3,5]}(\lambda; 1000)$](image-url)
we have
\[ g_1(\lambda) := \lim_{Q \to \infty} g_1^{[a,b]}(\lambda; Q) \]
\[ = \begin{cases} \frac{\pi^2}{\lambda^2} - \lambda, & 0 < \lambda < 1 \\ 6 \pi^2 & 1 < \lambda < 2 \\ -1 + \left( \frac{\lambda}{2} - \frac{3}{2} \right) \log \left( \frac{\lambda}{\lambda-1} \right) + \frac{3 \lambda}{2} \log \left( \frac{\lambda}{\lambda-1} \right), & 2 < \lambda \end{cases} \]

where the Dilogarithm is defined for \(|z| \leq 1\) by
\[ \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \]

Moreover, the error satisfies:
\[ \left| g_1^{[a,b]}(\lambda; Q) - g_1(\lambda) \right| \ll \epsilon (1 + \lambda) Q^{e-1/2} |b-a|. \]

More generally, we consider the joint distribution \(g_k(\lambda_1, \lambda_2, \ldots, \lambda_k)\), which describes the average distribution of \(k\)-tuples of consecutive gaps over the interval \([a,b]\), defined as
\[ g_k(\lambda_1, \lambda_2, \ldots, \lambda_k) = \lim_{Q \to \infty} g_k^{[a,b]}(\lambda_1, \ldots, \lambda_k; Q), \quad (1.2) \]

where
\[ g_k^{[a,b]}(\lambda_1, \ldots, \lambda_k; Q) = \frac{1}{Q} \int_a^b \frac{\# \{ \gamma_1 \cdots \gamma_k \in G_{Q,k}(\alpha) : \forall i \leq k, \gamma_i \geq \frac{\lambda_i}{Q} \} \} d\alpha. \]

When \(Q\) is taken to be 1000 the resulting graph is show in Figure 2. It can be proved that the functions \(g_k(\lambda_1, \ldots, \lambda_k)\) exist and are independent of the interval \([a,b]\) on which the average is taken. Towards the end of this paper we concentrate in this multidimensional case and find an explicit formula for \(g_2(\lambda_1, \lambda_2)\). We will deduce that the joint distribution \(g_2(\lambda_1, \lambda_2)\) is piecewise linear for \(\lambda_1 + \lambda_2 < 1\), that is,
\[ g_2(\lambda_1, \lambda_2) = 1 - \frac{6}{\pi^2} \text{Max}(\lambda_1, \lambda_2) - \frac{3}{\pi^2} \text{Min}(\lambda_1, \lambda_2), \quad \text{for } \lambda_1 + \lambda_2 < 1. \quad (1.3) \]

From a probabilistic point of view our questions of interest can be seen as follows. For a large \(Q\), consider the set \(S_Q(\alpha)\) whose average gap size is \(\frac{1}{Q}\). In the one dimensional case this is our main question: For a given \(\lambda_1\), what is the probability that a randomly selected gap will be greater than \(\lambda_1\) times the average gap? And for the the \(k\) dimensional case (say \(k = 2\)) the question is the following: for given \(\lambda_1\) and \(\lambda_2\) what is the probability that a gap and its neighbor to the right are both greater than the average times \(\lambda_1\) and \(\lambda_2\) respectively? Several papers have pointed out this interpretation of the problems of our interest (see [25], [26] and the reference there in). Others have also focused on the pair correlation of the sequence of fractional parts and similar aspects from
a random point of view (see [30], [33] and their references). We answer the two questions posed above by giving the cumulative distribution function for the 1-dimensional and 2-dimensional case. We would expect that in the $k$-dimensional case the events are not independent. This is made evident by the fact that $g(\lambda_1, \lambda_2)$ are not independent is not the product of $g(\lambda_1)$ and $g(\lambda_2)$ (by Theorem 1.1 and (1.3)). The proof of these theorems have three key stages which will determine the structure of the present paper. The next three sections of this paper are devoted to the one dimensional case $g(\lambda_1)$. We first establish a key connection of the elements of the sequence of fractional parts with Farey fractions in Section 2. The third section of the paper is devoted to some lemmas that allow us to write a sum over Farey fractions in a given region as an integral that is easier to handle. The main ingredients of these lemmas are Kloosterman sums. In the third section we write $g(\lambda_1)$ as a suitable sum for which we can apply the lemmas from the third section. After some other miscellaneous work we complete the proof of Theorem 1.1. The last two sections of this paper are devoted to the joint distribution from $k$-dimensional spacing between consecutive elements of the sequence of fractional parts, with special attention to $k = 2$.

We now finish this introduction with the following illustration of the three gaps theorem. Take $\alpha = \sqrt{2}$ and $Q = 10$. The points $S_{10}(\sqrt{2})$ along with the three gaps labeled $A$, $B$, and $C$ are shown in Figure 3. Let $G_{Q,k}(\alpha)$ denote the list of $k$-sequences of consecutive gaps in $S_Q(\alpha)$. When $k > 1$ we allow the sequence to wrap around 0 so that exactly $Q$ tuples of gaps are considered. For
example,
\[ G_{10,1}(\sqrt{2}) = \{A, C, A, B, A, C, A, B, A, B\}, \]
\[ G_{10,2}(\sqrt{2}) = \{AC, CA, AB, BA, AC, CA, AB, BA, AB, BA\}, \]
\[ G_{10,3}(\sqrt{2}) = \{ACA, CAB, ABA, BAB, ACA, CAB, ABA, BAB, ABA, BAC\}. \]

Also, in this case the permutation \( \sigma \) satisfies
\[ \{\sigma_n\}_{n=0}^9 = \{0, 5, 3, 8, 1, 6, 4, 9, 2, 7\}. \]

2 Key Connections to Farey Fractions

In order to understand the limit given in definition (1.1) we must first understand the cardinality of the set in the integrand. Specifically, given that we have three possible gaps sizes \( A, B \) and \( C \), we need to have more information on the number of gaps of each size for any given \( Q \), as well as a way to handle the contribution of each gap. This is accomplished in Lemma 2.1, where we identify the two smaller gaps \( A \) and \( B \) with the closest Farey fractions on either side of \( \alpha \). This will allow us to rewrite the integral from Theorem 1.1 in a form more useful for computations.

Lemma 2.1. For an irrational \( \alpha \), the lengths of three gaps \((A, B, \text{ and } C)\) generated by the sequence \( \{n\alpha\}_{0 \leq n < Q} \) may be computed as follows: Let us arrange the Farey fractions \( \{0 \leq \frac{a}{q} \leq 1 | (a,q) = 1, q < Q\} \) in order on the interval \([0,1]\) and choose consecutive fractions \( \frac{a_1}{q_1} \) and \( \frac{a_2}{q_2} \) so that
\[
\frac{a_1}{q_1} < \alpha < \frac{a_2}{q_2}.
\]

Then, the three gaps that can appear have lengths
\[
A = q_1\alpha - a_1 = \{q_1\alpha\}, \\
B = a_2 - q_2\alpha = 1 - \{q_2\alpha\}, \\
C = A + B,
\]

and the function \( \sigma \), which is a permutation of the set \( \{0,1,\ldots,Q-1\} \) so that the sequence
\[
\{\sigma_0\alpha\}, \{\sigma_1\alpha\}, \ldots, \{\sigma_{Q-1}\alpha\}
\]
is in increasing order, satisfies:
\[
\sigma_0 = 0, \\
\sigma_{i+1} - \sigma_i = \begin{cases} 
q_1, & \text{if } \sigma_i \in [0, Q - q_1) \quad (A \text{ Gap}) \\
q_1 - q_2, & \text{if } \sigma_i \in (Q - q_1, q_2) \quad (C \text{ Gap}) \\
-q_2, & \text{if } \sigma_i \in [q_2, Q) \quad (B \text{ Gap}) 
\end{cases}
\]

Remark 2.2. This Lemma gives a simple formula for the number of gaps of each size. The number of a gaps of each size is number of integers in the intervals on the right hand side of the recurrence relation for \( \sigma_i \). Thus, the numbers of \( A \) gaps, \( B \) gaps and \( C \) gaps are, respectively,
\[
Q - q_1, \quad Q - q_2, \quad q_1 + q_2 - Q.
\]
This is in agreement with the fact that the total number of gaps is $Q$ and the total length of these gap is 1 since

$$(Q - q_1) + (Q - q_2) + (q_1 + q_2 - Q) = Q,$$

$$(Q - q_1)A + (Q - q_2)B + (q_1 + q_2 - Q)C = 1.$$ 

This last equality is equivalent to the determinant property of Farey fractions:

$$a_2q_1 - a_1q_2 = 1.$$ 

Indeed, this property may be viewed as a corollary of Lemma 2.1

Remark 2.3. In [21] O’Bryant developed detailed account of permutations that order fractional parts. Here we have use an independent and simple idea to describe this permutations $\sigma_i$.

Proof. Let us define a shifting operation that will associate each gap in the sequence with the two gaps that have 0 (or 1) as an end point or with the sum of these two gaps. Let the $A$ gap be the first gap, that is, the one with 0 as a left end point, and let the $B$ gap be the last gap, that is, the gap with 1 as a right end point. Suppose that when the points $\{n\alpha\}$ for $0 \leq n < Q$ are arranged in order, $\{n_1x\}$ and $\{n_2x\}$ appear consecutively so that the interval $[\{n_1x\}, \{n_2x\}]$ is a gap. If either $n_1$ or $n_2$ are 0 then the gap is already the $A$ gap or the $B$ gap. If neither $n_1$ nor $n_2$ is 0, the gap $[\{n_1x\}, \{n_2x\}]$ has the same length as the interval $[\{(n_1 - 1)\alpha\}, \{(n_2 - 1)\alpha\}]$ (if $\{(n_1 - 1)\alpha\} > \{(n_2 - 1)\alpha\}$, we consider this interval as wrapping around through 0). This interval is also a gap except in exactly one case: the case when the point $\{Q\alpha\}$ lies in the gap $[\{n_1x\}, \{n_2x\}]$. In this case the gap $[\{n_1x\}, \{n_2x\}]$ is associated with the two smaller gaps $[\{(n_1 - 1)\alpha\}, \{(Q - 1)\alpha\}]$ and $[\{(Q - 1)\alpha\}, \{(n_2 - 1)\alpha\}]$. This proves that when an arbitrary starting gap is chosen, the shifting process will divide the starting gap into two smaller gaps at most once and that these smaller gaps will have the same size as the $A$ gap or the $B$ gap since this is where the shifting process terminates.

We now know that the size of the $A$ gap is

$$A = \min_{0 < n < Q} \{n\alpha\},$$

and the size of the $B$ gap is

$$B = 1 - \max_{0 < n < Q} \{n\alpha\}.$$ 

We need to show that the minimum and maximum are attained at $n = q_1$ and $n = q_2$ respectively, where $q_1$ and $q_2$ are as in the statement of the lemma. Suppose there is another $q < Q$ with $\{q\alpha\} < \{q_1\alpha\}$ and write $\{q\alpha\} = q\alpha - a$. Then

$$0 < q\alpha - a < q_1\alpha - a_1.$$ 

(2.1)

Since $\frac{a_1}{q_1}$ is the greatest Farey fraction less than $\alpha$, we have

$$\frac{a}{q} < \frac{a_1}{q_1} < \alpha.$$ 

In the case $q_1 > q$, we have the trivial inequality

$$\frac{a_1}{q_1} < \frac{a_1 - a}{q_1 - q}$$
from which deduce that, since \( \frac{a_1}{q_1} \) is the greatest Farey fraction less than \( \alpha \),

\[
\alpha < \frac{a_1 - a}{q_1 - q},
\]

which contradicts (2.1). In the case \( q_1 < q \), we have the similar inequality

\[
\frac{a - a_1}{q - q_1} < \frac{a_1}{q_1},
\]

from which we deduce that

\[
\frac{a - a_1}{q - q_1} < \alpha,
\]

which also contradicts (2.1). This proves that the minimum is attained when \( n = q_1 \). The fact that the maximum is attained at \( n = q_2 \) follows by symmetry since

\[1 - \max_{0 < n < Q} \{ n\alpha \} = \min_{0 < n < Q} \{ n(1 - \alpha) \}.\]

Now suppose that \( \{\sigma_i\alpha\} \) is a point of \( S_Q(\alpha) \). If \( \sigma_i \in [0, Q - q_1) \), then the \( A \) gap, which is the interval

\[\{0, q_1\alpha\},\]

can be shifted to the interval

\[\{\{\sigma_i\alpha\}, \{\sigma_i + q_1\alpha\}\},\]

so that \( \sigma_{i+1} = \sigma_i + q_1 \) in this case. In the case \( \sigma_i \in [q_2, Q) \), the gap between \( \sigma_i \) and \( \sigma_{i+1} \) is a \( B \) gap, so \( \sigma_{i+1} = \sigma_i - q_2 \) in this case. Finally, when \( \sigma_i \in [Q - q_1, q_2) \), the gap is a \( C \) gap and so \( \sigma_{i+1} = \sigma_i + q_1 - q_2 \).

\[\square\]

3 A Lemma Via Kloosterman Sums

We will transform our integral into a sum of integrals over Farey arcs in a certain region. The resulting sum depends only on the denominators of the Farey fractions. This fact is crucial and explains the heuristic reason why the limit approaches the same distribution function in any interval. We present a general result for any short interval by borrowing Lemma 8 from [4] (We have normalized the function and region by a factor of \( Q \)).

Lemma 3.1. Let

\[
\sum_{\Omega} f := \sum_{q_1, q_2} f \quad \text{where} \quad \frac{q_1}{Q}, \frac{q_2}{Q} \in \Omega, \quad a \leq \frac{q_1}{q} < \frac{q_2}{Q} \leq b
\]

Then if \( \Omega \) is convex subregion of \([0, 1] \times [0, 1]\) with rectifiable boundary, and \( f \) is a \( C^1 \) function on \( \Omega \), we have

\[
\left| \frac{1}{b - a} \sum_{\Omega} f \left( \frac{q_1}{Q}, \frac{q_2}{Q} \right) \frac{1}{Q^2} \int \int_{\Omega} f(x, y) dx dy \right| \ll \frac{1}{b - a} \left( \left\| \frac{\partial f}{\partial x} \right\|_{\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{\infty} \right) \frac{\text{Area}(\Omega) \log Q}{Q} + \frac{\|f\|_{\infty} \left(1 + \log(\partial \Omega) \right) \log Q}{Q} + \frac{m_f \|f\|_{\infty}}{Q^{1/2 - \epsilon}},
\]

where \( m_f \) is an upper bound for the number of intervals of monotonicity of each of the maps \( y \to f(x, y) \).
4 Proof of Theorem 1.1

We will handle the contribution of the three gaps of each type A, B, and C separately by partitioning the interval \([a, b]\) into Farey arcs with denominators strictly less than \(Q\). Lemma 3.1 will then enable us to calculate the limit as \(Q \to \infty\). We will deal with the error terms when we prove a general result for \(g_k\) in the next section. In the interval \([\frac{a_1}{q_1}, \frac{a_2}{q_2}]\), the A gaps contribute the amount

\[
\int_{\frac{a_1}{q_1}}^{\frac{a_2}{q_2}} \frac{1}{Q} \left\{ Q - q_1, \quad q_1 \alpha - a_1 \geq \frac{\lambda}{Q} \right\} d\alpha,
\]

since the A gaps are \(Q - q_1\) in number by Lemma 2.1. If a new integration variable \(t\) defined by

\[
\alpha = \frac{a_1}{q_1} + \frac{t}{q_1 q_2}
\]

is introduced, the integral becomes

\[
\int_0^1 \frac{dt}{Q q_1 q_2} \left\{ \frac{Q - q_1}{q_1 q_2} \left( 1 - \frac{\lambda q_2}{Q} \right), \quad 0 \leq \frac{\lambda q_2}{Q} \leq 1 \right\},
\]

To compute the total contribution of the A gaps to the function \(g_1(\lambda)\) we need to sum this expression over consecutive pairs of Farey fractions in the interval \([a, b]\) as

\[
\frac{1}{b - a} \sum_{a \leq q_1 < q_2 \leq b} \left\{ \frac{Q - q_1}{q_1 q_2} \left( 1 - \frac{\lambda q_2}{Q} \right), \quad 0 \leq \frac{\lambda q_2}{Q} \leq 1 \right\}.
\]

By Lemma 3.1, this sum converges to the integrals

\[
\frac{6}{\pi^2} \left\{ \begin{array}{cl} \int_0^1 \int_0^1 \frac{(1-x)(1-\lambda y)}{xy} dx dy, & 0 \leq \lambda \leq 1 \\ \int_0^{\lambda} \frac{1}{x} \int_0^1 \frac{(1-x)(1-\lambda y)}{xy} dx dy, & 1 < \lambda \end{array} \right.,
\]

which further equals

\[
\frac{6}{\pi^2} \left\{ \begin{array}{cl} -1 - \frac{1}{2} \lambda + \frac{\pi^2}{6}, & 0 \leq \lambda \leq 1 \\ -1 - \frac{1}{2} \lambda + (1 - \lambda) \log \left( 1 - \frac{1}{\lambda} \right) + \text{Li}_2 \left( \frac{1}{\lambda} \right), & 1 < \lambda \end{array} \right.,
\]

and this is the contribution of the A gaps to the function \(g_1(\lambda)\). The B gaps contribute the amount

\[
\int_{\frac{a_1}{q_1}}^{\frac{a_2}{q_2}} \frac{1}{Q} \left\{ Q - q_2, \quad a_2 - q_2 \alpha \geq \frac{\lambda}{Q} \right\} d\alpha
\]

in the interval \([\frac{a_1}{q_1}, \frac{a_2}{q_2}]\) since they are \(Q - q_2\) in number. The calculations to complete the total contribution of the B gaps are similar to those of the A gaps, and the total contribution is found
to be the same as (4.2). By Lemma 2.1, the \( C \) gaps are \( q_1 + q_2 - Q \) in number, and thus contribute

\[
\int_{\frac{q_1}{q_1}}^{\frac{q_2}{q_2}} \frac{1}{Q} \begin{cases} 
q_1 + q_2 - Q, & (q_1 \alpha - a_1) + (a_2 - q_2 \alpha) \geq \lambda/Q \\
0, & (q_1 \alpha - a_1) + (a_2 - q_2 \alpha) < \lambda/Q 
\end{cases} \ d\alpha
\]

in the interval \( [\frac{q_1}{q_1}, \frac{q_2}{q_2}] \). With the substitution (4.1), this integral becomes

\[
\int_0^1 \frac{dt}{Nq_1q_2} \begin{cases} 
q_1 + q_2 - N, & \frac{1-t}{q_1} + \frac{t}{q_2} \geq \frac{\lambda}{N} \\
0, & \frac{1-t}{q_1} + \frac{t}{q_2} < \frac{\lambda}{N}
\end{cases}
\]

where the numerators \( a_1 \) and \( a_2 \) of the Farey fractions have conveniently canceled out. If \( q_2 < q_1 \), this last integral has the evaluation

\[
\begin{cases}
q_1 + q_2 - N, \\
q_1 + q_2 - N \frac{q_1(N - \lambda q_2)}{N(q_1 - q_2)}, \\
0,
\end{cases}
\]

\[
\begin{array}{l}
q_1 N < \frac{\lambda}{X} \\
q_2 N < \frac{1}{X} < \frac{q_1}{N},
\end{array}
\]

while if \( q_1 < q_2 \), the substitution \( t \to 1 - t \) shows that the integral has the same evaluation with \( q_1 \) and \( q_2 \) switched:

\[
\begin{cases}
q_1 + q_2 - N, \\
q_1 + q_2 - N \frac{q_2(N - \lambda q_1)}{N(q_2 - q_1)}, \\
0,
\end{cases}
\]

\[
\begin{array}{l}
q_2 N < \frac{\lambda}{X} \\
q_1 N < \frac{1}{X} < \frac{q_2}{N},
\end{array}
\]

Thus, the total contribution of the \( C \) gaps in the interval \( [a, b] \) is

\[
\frac{1}{b-a} \sum_{a \leq \frac{q_1}{q_1} < \frac{q_2}{q_2} \leq b} \begin{cases}
q_1 + q_2 - N, \\
q_1 + q_2 - N \frac{q_1(N - \lambda q_2)}{N(q_1 - q_2)}, \\
0,
\end{cases}
\]

\[
\begin{array}{l}
q_1 N < \frac{\lambda}{X} \\
q_2 N < \frac{1}{X} < \frac{q_1}{N},
\end{array}
\]

\[
\frac{1}{b-a} \sum_{a \leq \frac{q_1}{q_1} < \frac{q_2}{q_2} \leq b} \begin{cases}
q_1 + q_2 - N, \\
q_1 + q_2 - N \frac{q_2(N - \lambda q_1)}{N(q_2 - q_1)}, \\
0,
\end{cases}
\]

\[
\begin{array}{l}
q_2 N < \frac{\lambda}{X} \\
q_1 N < \frac{1}{X} < \frac{q_2}{N},
\end{array}
\]

By Lemma 3.1, the first of these sums approaches the integrals

\[
6 \pi^2 \begin{cases}
\int \int_{\frac{1}{2}}^{1} \frac{x+y-1}{xy} dydx, & 0 \leq \lambda \leq 1 \\
\int \int_{\frac{1}{2}}^{1} x+y-1 xy dydx + \int \int_{\frac{1}{2}}^{1} \frac{x+y-1 - \lambda y}{y} x-y dydx, & 1 \leq \lambda \leq 2 \\
\int \int_{0}^{1} x+y-1 \frac{\lambda y}{y} x-y dydx, & 2 \leq \lambda
\end{cases}
\]

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and the second sum approaches the same set of integrals after the change of variable \( x \leftrightarrow y \) is made. Now, these integrals can also be evaluated with the Dilogarithm, and the result is the expression

\[
\frac{6}{\pi^2} \begin{cases}
\frac{1}{2} - \frac{\pi^2}{6} + \frac{\log^2(2)}{2} + \frac{1}{2 \lambda} \log \left( \frac{2}{\lambda} - 1 \right) - \frac{1}{\lambda} \log \left( \frac{2-\lambda}{\lambda-1} \right) - \log \left( \frac{\lambda-1}{\lambda} \right), & 0 \leq \lambda \leq 1 \\
+ \frac{1}{2} \log \left( \frac{\lambda}{2} \right) \log(\lambda) + \text{Li}_2 \left( \frac{\lambda}{2} \right) + \text{Li}_2 \left( \frac{\lambda}{2} \right), & 1 < \lambda < 2 \\
\frac{3}{4} - \frac{\pi^2}{12} - \frac{\log^2(2)}{2} + \frac{\log(2)}{2}, & \lambda = 2 \\
\frac{1}{2} + \frac{1}{2 \lambda} + \frac{1}{4} \log \left( \frac{1-2 \lambda}{2} \right) - \log \left( 1 - \frac{1}{\lambda} \right) + \frac{1}{\lambda} \log \left( \frac{\lambda-1}{\lambda-2} \right) + \text{Li}_2 \left( \frac{1}{\lambda} \right) - \text{Li}_2 \left( \frac{2}{\lambda} \right), & 2 < \lambda
\end{cases}
\] (4.3)

for the contribution of the first sum to the function \( g_1(\lambda) \). Thus, \( g_1(\lambda) \) is the twice the sum of (4.3) and (4.2). Since

\[
\text{Li}_2(z) = -\frac{\log(1-z)}{z},
\]

it is now straightforward to calculate \( g_1'(\lambda) \) and verify that it is differentiable everywhere, that is, the following integral representation is valid:

\[
g_1(\lambda) = \int_0^1 -g_1' \left( \frac{1}{x} \right) \frac{dx}{x^2},
\]

where

\[-g_1' \left( \frac{1}{x} \right) = \frac{6}{\pi^2} \begin{cases}
x + \log \frac{|1-x|^2(1-x)^2}{|1-2x|^2(1-2x)^2}, & 0 < x < 1 \\
1, & 1 < x
\end{cases}.
\]

5 The Joint Distribution from \( k \)-Dimensional Spacing

We now focus in generalizing Theorem 1.1, but not by introducing a \( k \)-tuple of \( \alpha \)'s but instead we consider a \( k \)-tuple of \( \lambda \)'s as in definition 1.2. In the previous theorem two key properties were used: the fact that the number of gaps of each of the three sizes is a simple function of \( q_1 \), and \( q_2 \), and the fact that all of the numerators of the Farey fraction canceled out of the final calculations. The exact same phenomenon happens in the general case for the function \( g_k \). An exact statement about the number of times a given sequence of consecutive gap sizes appears is given in the following Lemma. Set

\[
||I|| = \text{Length}(I \cap [0,1]).
\]

**Lemma 5.1.** For any sequence \( \gamma_1 \gamma_2 \cdots \gamma_k \) of the gap sizes \( A, B, \) and \( C \), there is a continuous function

\[
f_{\gamma_1 \gamma_2 \cdots \gamma_k}(x,y) : T \rightarrow [0,1]
\]

such that for \( q_1 \) and \( q_2 \) as in Lemma 2.1,

\[
\frac{\text{#}\{\Gamma \in G_{Q,k}(\alpha) : \Gamma = \gamma_1 \gamma_2 \cdots \gamma_k\}}{Q} = f_{\gamma_1 \gamma_2 \cdots \gamma_k} \left( \frac{q_1}{Q}, \frac{q_2}{Q} \right).
\]

That is, the average number of times the sequence \( \gamma_1 \gamma_2 \cdots \gamma_k \) appears as a sequence of consecutive gap sizes in \( S_Q(\alpha) \) may be interpolated by a continuous function uniformly in \( Q \).
Remark 5.2. By Remark 2.2, we have
\[ f_A(x, y) = 1 - x, \]
\[ f_B(x, y) = 1 - y, \]
\[ f_C(x, y) = x + y - 1. \]

Proof. By Lemma 2.1, we may write \( f_{\gamma_1 \gamma_2 \cdots \gamma_k} \left( \frac{a}{Q}, \frac{b}{Q} \right) \) as the length of the intersection of \( k \) intervals whose endpoints are continuous functions of \( q_1 \) and \( q_2 \). The general truth of this fact will be evident from the proof of the particular case \( \gamma_1 \gamma_2 \cdots \gamma_k = ABC \), for example. By Lemma 2.1 we have
\[
\frac{\# \{ \Gamma \in G_{Q,3}(\alpha) : \Gamma = ABC \}}{Q} = \frac{1}{Q} \sum_{0 \leq i < Q} \left\{ \begin{array}{l}
\{ \sigma_{i+1} \alpha \} - \{ \sigma_{i+0} \alpha \} = A, \\
\{ \sigma_{i+2} \alpha \} - \{ \sigma_{i+1} \alpha \} = B, \\
\{ \sigma_{i+3} \alpha \} - \{ \sigma_{i+2} \alpha \} = C
\end{array} \right\}
\]
\[
= \frac{1}{Q} \sum_{0 \leq \sigma < Q} \left\{ \begin{array}{l}
\sigma \in [0, Q - q_1), \\
\sigma + q_1 \in [q_2, Q), \\
\sigma + q_1 - q_2 \in [Q - q_1, q_2]
\end{array} \right\}
\]
\[
= \frac{1}{Q} \left\| [0, Q - q_1) \cap [q_2 - q_1, Q - q_1) \cap [Q + q_2 - 2q_1, 2q_1 - q_1] \right\|
\]
In this case, we then have
\[
f_{ABC}(x, y) = \left\| [0, 1 - x) \cap [y - x, 1 - x) \cap [1 + y - 2x, 2x - y] \right\|,
\]
which is clearly a continuous function. \( \Box \)

Theorem 5.3. For any integer \( k \), the limit
\[
g_k(\lambda_1, \ldots, \lambda_k) = \lim_{Q \to \infty} g_k^{[a,b]}(\lambda_1, \ldots, \lambda_k; Q)
\]
\[
= \lim_{Q \to \infty} \frac{1}{b-a} \int_a^b \frac{\# \{ \gamma_1 \cdots \gamma_k \in G_{Q,k}(\alpha) : \forall i \leq k \gamma_i \geq \frac{\lambda_i}{Q} \}}{Q} d\alpha
\]
exists and is independent of the interval \([a, b]\) on which the average is taken. A formula for \( g_k \) and an error estimate is given by:
\[
\left| g_k^{[a,b]}(\lambda_1, \ldots, \lambda_k; Q) - \frac{1}{\zeta(2)} \sum_{\gamma_1 \cdots \gamma_k \in \{A,B,C\}^k} \int_T f_{g_1 \cdots g_k}(x, y) |\chi_{g_1}(\lambda_1) \cap \cdots \cap \chi_{g_k}(\lambda_k)| \frac{dxdy}{xy} \right|
\]
\[
\ll_k \epsilon (1 + \lambda_1) \cdots (1 + \lambda_k) \frac{Q^{k-1/2}}{|b-a|}
\]
where
\[
\chi_A(\lambda) = \chi_A(x, y, \lambda) = [\lambda y, \infty)
\]
\[
\chi_B(\lambda) = \chi_B(x, y, \lambda) = (-\infty, 1 - \lambda x]
\]
\[
\chi_C(\lambda) = \chi_C(x, y, \lambda) = \left\{ \begin{array}{l}
\frac{y(\lambda x - 1)}{x-y}, \infty) \\
(-\infty, 1 - \frac{x(\lambda y - 1)}{y-x})
\end{array} \right\}
\]
\[
y < x \]
and $f_{\gamma_1, \ldots, \gamma_k}(x, y)$ is defined in Lemma 5.1. The function $g_k(\lambda_1, \ldots, \lambda_k)$ also satisfies
\[ g_k(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k) = g_k(\lambda_k, \lambda_{k-1}, \ldots, \lambda_2, \lambda_1). \]

**Proof.** We first divide the integral defining $g_k$ along the Farey fractions in $[a, b]$ with denominator strictly less than $Q$:
\[
g_k^{[a,b]}(\lambda_1, \ldots, \lambda_k; q) = \frac{1}{b-a} \sum_{a \leq \frac{a_1}{q_1}, \frac{a_2}{q_2} \leq b} \frac{1}{Q} \int_{\frac{a_1}{q_1}}^{\frac{a_2}{q_2}} \# \{ \gamma_1 \cdots \gamma_k \in G_{Q,k}(\alpha) : \forall i \leq k \gamma_i \geq \frac{\lambda_i}{Q} \} d\alpha.
\]

We next divide up the gap sequences among all of the $3^k$ possibilities:
\[
= \sum_{\gamma_1 \cdots \gamma_k \in \{A,B,C\}^k} \frac{1}{b-a} \sum_{a \leq \frac{a_1}{q_1}, \frac{a_2}{q_2} \leq b} \frac{1}{Q} \int_{\frac{a_1}{q_1}}^{\frac{a_2}{q_2}} \frac{1}{Q} \# \{ \Gamma \in G_{Q,k}(\alpha) : \Gamma = \gamma_1 \cdots \gamma_k \}
\times \begin{cases} 
1, & \forall i \leq k \gamma_i \geq \frac{\lambda_i}{Q} \\
0, & \exists i \leq k \gamma_i < \frac{\lambda_i}{Q}
\end{cases} d\alpha.
\]

The conditions that an $A$, $B$, or $C$ gap is bigger than $\frac{\lambda}{Q}$ are by Lemma 2.1:
\[
q_1 \alpha - a_1 \geq \frac{\lambda}{Q}, \quad \text{(for an A gap)}
\]
\[
a_2 - q_2 \alpha \geq \frac{\lambda}{Q}, \quad \text{(for a B gap)}
\]
\[
q_1 \alpha - a_1 + a_2 - q_2 \alpha \geq \frac{\lambda}{Q}, \quad \text{(for a C gap)}
\]

Under the change of variable
\[
\alpha = \frac{a_1}{q_1} + \frac{t}{q_1 q_2}
\]

these conditions become
\[
t \geq \frac{\lambda q_2}{Q}, \quad \text{(for an A gap)}
\]
\[
t \leq 1 - \frac{\lambda q_1}{Q}, \quad \text{(for a B gap)}
\]
\[
(q_1 - q_2) t \geq \frac{\lambda q_1 q_2}{Q} - q_2, \quad \text{(for a C gap)}
\]

respectively. Thus, we have
\[
\int_{\frac{a_1}{q_1}}^{\frac{a_2}{q_2}} \frac{1}{Q} \# \{ \Gamma \in G_{Q,k}(\alpha) : \Gamma = \gamma_1 \cdots \gamma_k \} \begin{cases} 
1, & \forall i \leq k \gamma_i \geq \frac{\lambda_i}{Q} \\
0, & \exists i \leq k \gamma_i < \frac{\lambda_i}{Q}
\end{cases} d\alpha
\]
\[
= f_{\gamma_1, \ldots, \gamma_k} \left( \frac{q_1}{Q}, \frac{q_2}{Q} \right) \left| \chi_{\gamma_1} \left( \frac{q_1}{Q}, \frac{q_2}{Q}, \lambda_1 \right) \cap \cdots \cap \chi_{\gamma_k} \left( \frac{q_1}{Q}, \frac{q_2}{Q}, \lambda_k \right) \right| \frac{1}{q_1 q_2}
\]
So, finally,
\[ g_k^{[a,b]}(\lambda_1, \ldots, \lambda_k; Q) = \frac{1}{b - a} \sum_{T} f\left( \frac{q_1}{Q}, \frac{q_2}{Q} \right) \frac{1}{Q^2}, \]
where
\[ f(x, y) = f_{\gamma_1 \cdots \gamma_k}(x, y) \left| \frac{|X_{\gamma_1}(x, y, \lambda_1) \cap \cdots \cap X_{\gamma_k}(x, y, \lambda_k)|}{xy} \right|, \]
and \( T \) is the triangle \( \{(x, y) : x \leq 1, y \leq 1, x + y \geq 1\} \). In order to apply Lemma 3.1, we need bounds for \( f \) and its partial derivatives. Since \( f \) is unbounded and its partials are unbounded on \( T \), the domain must be shrunk to one of the form
\[ T_\delta = \{(x, y) : x \leq 1, y \leq 1, x + y \geq 1 + \delta, |x - y| > \delta\}, \]
and set \( \|\cdot\|_\infty \) to be the supremum norm on \( T_\delta \). Now, the functions \( f_{\gamma_1 \cdots \gamma_k} \) are relatively tame; they satisfy
\[ \|f_{\gamma_1 \cdots \gamma_k}\|_\infty \leq 1, \]
\[ \left\| \frac{\partial f_{\gamma_1 \cdots \gamma_k}}{\partial x} \right\|_\infty + \left\| \frac{\partial f_{\gamma_1 \cdots \gamma_k}}{\partial y} \right\|_\infty \leq 2k. \]
Hence we focus on the functions
\[ F_{\gamma_1 \cdots \gamma_k}(x, y) = \left| \frac{|X_{\gamma_1}(x, y, \lambda_1) \cap \cdots \cap X_{\gamma_k}(x, y, \lambda_k)|}{xy} \right|. \]
Since \( 0 \leq F_{\gamma_1 \cdots \gamma_k}(x, y) \leq \frac{1}{xy} \),
\[ \|F_{\gamma_1 \cdots \gamma_k}\|_\infty \leq \frac{1}{\delta}. \]
Next, the fact that
\[ \left\| \frac{\partial F_{\gamma_1 \cdots \gamma_k}}{\partial x} \right\|_\infty + \left\| \frac{\partial F_{\gamma_1 \cdots \gamma_k}}{\partial y} \right\|_\infty \ll k \frac{\lambda_1 \cdots \lambda_k}{\delta^2} \]
follows from the following table:

<table>
<thead>
<tr>
<th>( h(x, y) )</th>
<th>bound for ( |h|_\infty )</th>
<th>bound for ( \left| \frac{\partial h}{\partial x} \right|<em>\infty + \left| \frac{\partial h}{\partial y} \right|</em>\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>X_A(x, y, \lambda)</td>
<td>)</td>
</tr>
<tr>
<td>(</td>
<td>X_B(x, y, \lambda)</td>
<td>)</td>
</tr>
<tr>
<td>(</td>
<td>X_C(x, y, \lambda)</td>
<td>)</td>
</tr>
<tr>
<td>( \frac{1}{xy} )</td>
<td>( \frac{1}{\delta} )</td>
<td>( \frac{1}{\delta^2} )</td>
</tr>
</tbody>
</table>

In summary, we have
\[ \|f\|_\infty \ll_k \frac{1}{\delta}, \]
\[ \left\| \frac{\partial f}{\partial x} \right\|_\infty + \left\| \frac{\partial f}{\partial y} \right\|_\infty \ll_k \frac{\lambda_1 \cdots \lambda_k}{\delta^2}. \]
Therefore, by Lemma 3.1 with $m = O_k(1)$,

\[
\left| \frac{1}{b-a} \sum_{T_s} f \left( \frac{q_1}{Q}, \frac{q_2}{Q} \right) \frac{1}{Q^2} - \frac{1}{\zeta(2)} \int_{T_s} f(x,y) dxdy \right| \\
\ll_{k,\epsilon} \frac{1}{b-a} \left( \frac{\lambda_1 \cdots \lambda_k \log Q}{\delta^2 Q} + \frac{\log Q}{\delta Q} + \frac{Q^\epsilon}{\delta^{1/2} Q} \right) \\
\ll_{k,\epsilon} \frac{Q^{r-1}}{b-a} \left( \frac{\Lambda}{\delta^2} + Q^{1/2} \right),
\]

where $\Lambda = (1 + \lambda_k) \cdots (1 + \lambda_k)$. Finally,

\[
\left| \frac{1}{b-a} \sum_{T_s} f \left( \frac{q_1}{Q}, \frac{q_2}{Q} \right) \frac{1}{Q^2} - \frac{1}{\zeta(2)} \int_{T_s} f(x,y) dxdy \right| \\
\leq \left| \frac{1}{b-a} \sum_{T_s} f \left( \frac{q_1}{Q}, \frac{q_2}{Q} \right) \frac{1}{Q^2} - \frac{1}{\zeta(2)} \int_{T_s} f(x,y) dxdy \right| \\
+ \frac{1}{b-a} \left| \sum_{T\setminus T_s} f \left( \frac{q_1}{Q}, \frac{q_2}{Q} \right) \frac{1}{Q^2} \right| + \frac{1}{\zeta(2)} \int_{T\setminus T_s} f(x,y) dxdy \\
\ll_{k,\epsilon} \frac{Q^{r-1}}{b-a} \left( \frac{\Lambda}{\delta^2} + Q^{1/2} \right) + \frac{1}{b-a} \delta.
\]

This error term is optimized with the choice $\delta = Q^{-\epsilon}$, leading to

\[
\left| \frac{1}{b-a} \sum_{T_s} f \left( \frac{q_1}{Q}, \frac{q_2}{Q} \right) \frac{1}{Q^2} - \frac{1}{\zeta(2)} \int_{T_s} f(x,y) dxdy \right| \ll_{k,\epsilon} \frac{\Lambda Q^{r-1/2}}{b-a}.
\]

The symmetry of the function $g_k$ can be obtained from the symmetry of the function $g_k^{[0,1]}(\lambda_1, \ldots, \lambda_k; Q)$ as:

\[
g_k(\lambda_1, \ldots, \lambda_k; Q) = \lim_{Q \to \infty} g_k^{[0,1]}(\lambda_1, \ldots, \lambda_k; Q) \\
= \lim_{Q \to \infty} g_k^{[0,1]}(\lambda_k, \ldots, \lambda_1; Q) \\
= g_k(\lambda_k, \ldots, \lambda_1; Q).
\]

The function $g_k^{[0,1]}(\lambda_1, \ldots, \lambda_k; Q)$ has this symmetry property because the sequence of gaps in $S_Q(1-\alpha)$ is the reverse of the sequence of gaps in $S_Q(\alpha)$ with the $A$ and $B$ gaps switched.

6 Explicit Formula for $g_2(\lambda_1, \lambda_2)$

Theorem 5.3 gives a formula for $g_2(\lambda_1, \lambda_2)$ as an integral over the region $T$. By dividing up $[0, \infty) \times [0, \infty)$ into the 14 regions shown in Figure 4, we calculate $g_2(\lambda_1, \lambda_2)$ on each of the regions $A, B, C, D, E, F, \text{ and } G$. The resulting expressions are given in the following Theorem.
Figure 4: regions on which $g_2(\lambda_1, \lambda_2)$ is $C^\infty$ smooth
Theorem 6.1. The explicit formula for $g_2(\lambda_1, \lambda_2)$ on each of the regions $A, B, C, D, E, F,$ and $G$ in Figure 4 are as follows. The value of $g_2(\lambda_1, \lambda_2)$ on $A', B', C', D', E', F',$ and $G'$ may be found using the symmetry property

$$g_2(\lambda_1, \lambda_2) = g_2(\lambda_2, \lambda_1)$$

given in Theorem 5.3.

$$\frac{\pi^2}{6} g_2(\lambda_1, \lambda_2)|_A = \frac{\pi^2}{6} - \lambda_1 - \frac{\lambda_2}{2},$$

$$\frac{\pi^2}{6} g_2(\lambda_1, \lambda_2)|_B = \frac{\pi^2}{6} - 2 + \lambda_1 + \frac{3\lambda_2}{2} - 2 Li_2(\lambda_1) + 2 Li_2(1 - \lambda_2) + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \log(\lambda_1 + \lambda_2)$$

$$+ \left(\frac{1}{\lambda_1} - \lambda_1\right) \log\left(\frac{1 - \lambda_1}{\lambda_2}\right) + \log\left(\frac{1 - \lambda_2}{\lambda_1}\right) \left(-\lambda_2 + \frac{1}{\lambda_2} + 2 \log(\lambda_2)\right)$$

$$+ \log^2(\lambda_1) + (\lambda_2 - 2 \log(2\lambda_2)) \log(\lambda_1) + (2 \log(\lambda_1 - \lambda_2) - 2 \log(1 - \lambda_2)) \log(\lambda_2)$$

$$+ 2 Li_2\left(\frac{1}{\lambda_1}\right) + 2 Li_2\left(\frac{\lambda_1}{2}\right) - 2 Li_2(\lambda_2) + 2 Li_2\left(\frac{\lambda_2 - 1}{\lambda_2 - \lambda_1}\right) - 2 Li_2\left(\frac{\lambda_1(\lambda_2 - 1)}{\lambda_2 - \lambda_1}\right),$$

$$\frac{\pi^2}{6} g_2(\lambda_1, \lambda_2)|_C = \frac{3\lambda_2}{2} - \frac{\pi^2}{3} - 1 + \log^2(2)$$

$$+ \left(\frac{2}{\lambda_1} - 2\lambda_1\right) \log(\lambda_1 - 1) + \left(\frac{\lambda_1}{2} - \frac{2}{\lambda_1}\right) \log(2 - \lambda_1) + \left(\frac{\lambda_2}{2} - \frac{2}{\lambda_2}\right) \log(\lambda_2)$$

$$+ \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \log(\lambda_1 + \lambda_2) + \left(\lambda_1 - \lambda_2 + \frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) \log(\lambda_1 - \lambda_2)$$

$$+ \log^2(\lambda_1) + (\lambda_2 - 2 \log(2\lambda_2)) \log(\lambda_1) + (2 \log(\lambda_1 - \lambda_2) - 2 \log(1 - \lambda_2)) \log(\lambda_2)$$

$$+ 2 Li_2\left(\frac{1}{\lambda_1}\right) + 2 Li_2\left(\frac{\lambda_1}{2}\right) - 2 Li_2(\lambda_2) + 2 Li_2\left(\frac{\lambda_2 - 1}{\lambda_2 - \lambda_1}\right) - 2 Li_2\left(\frac{\lambda_1(\lambda_2 - 1)}{\lambda_2 - \lambda_1}\right),$$

$$\frac{\pi^2}{6} g_2(\lambda_1, \lambda_2)|_D = -\frac{5\pi^2}{6} - \frac{\lambda_1}{2} + \lambda_2 + \left(-\lambda_1 + \lambda_2 + \frac{2}{\lambda_1}\right) \log(2)$$

$$+ \left(3 - 3\lambda_1\right) \log(\lambda_1 - 1) + \left(\frac{2}{\lambda_1} - \frac{\lambda_1}{2} - 2 \log(2)\right) \log(\lambda_2) + \left(\lambda_1 - \frac{4}{\lambda_1}\right) \log(2 - \lambda_1)$$

$$+ 2 \left(\lambda_1 - \lambda_2 + \frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) \log(\lambda_1 - \lambda_2) + \left(\frac{\lambda_1}{2} - \lambda_2 - \frac{2}{\lambda_2}\right) \log(\lambda_1) + \left(\lambda_2 - \frac{1}{\lambda_2}\right) \log(1 - \lambda_2)$$

$$+ \log^2(\lambda_1) - 2 \log(-\lambda_1 + \lambda_2 + 2) \log(\lambda_1) + 2 \log(\lambda_1 - \lambda_2) \log(\lambda_2) + 2 \log(2) \log(2 - \lambda_1 + \lambda_2)$$

$$+ 4 Li_2\left(\frac{1}{\lambda_1}\right) + 2 Li_2\left(\frac{\lambda_1}{2}\right) + 2 Li_2(1 - \lambda_2) + 2 Li_2(\lambda_2) - 2 Li_2\left(\frac{2\lambda_2}{\lambda_1(2 - \lambda_1 + \lambda_2)}\right)$$

$$+ 2 Li_2\left(\frac{\lambda_2 - 1}{\lambda_2 - \lambda_1}\right) + 2 Li_2\left(\frac{\lambda_2}{2 - \lambda_1 + \lambda_2}\right)$$

$$- 2 Li_2\left(\frac{2 - \lambda_1 + \lambda_2}{2}\right) - 2 Li_2\left(\frac{\lambda_1 - \lambda_1 \lambda_2}{\lambda_1 - \lambda_2}\right),$$

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\[
\frac{\pi^2}{6} g_2(\lambda_1, \lambda_2)|_E = \frac{1}{6} (-3\lambda_1 - 2\pi^2 + 6) + \left( -\lambda_1 + \lambda_2 + \frac{2}{\lambda_1} \right) \log(2) - \log^2(2) + \log^2(\lambda_1) \\
+ \left( \frac{\lambda_1}{2} - \lambda_2 + 1 \frac{2}{\lambda_1} \right) \log(2\lambda_2 - \lambda_1) + \left( \frac{1}{\lambda_1} - \lambda_1 \right) \log(\lambda_1 - 1) \\
+ \left( \lambda_1 - \frac{2}{\lambda_1} \right) \log(2 - \lambda_1) + \left( \frac{1}{2} - \frac{1}{\lambda_2} \right) \log(\lambda_1) + \left( \lambda_2 - \frac{1}{\lambda_2} \right) \log(\lambda_2 - 1) \\
+ 2 \log(2) \log(2 - \lambda_1 + \lambda_2) - \log(-\lambda_1 + \lambda_2 + 1) + \log(\lambda_2) \left( -\frac{\lambda_1}{2} + \frac{2}{\lambda_1} - 2 \log(2) \right) \\
+ (2 \log(\lambda_2) + 2 \log(-\lambda_1 + \lambda_2 + 1) - 2 \log(-\lambda_1 + \lambda_2 + 2) - 2 \log(2\lambda_2 - \lambda_1)) \log(\lambda_1) \\
+ 2 \text{Li}_2 \left( \frac{1}{\lambda_1} \right) + 2 \text{Li}_2 \left( \frac{\lambda_1}{2} \right) - 2 \text{Li}_2 \left( \frac{\lambda_1 - 2\lambda_2}{2(\lambda_1 - \lambda_2 - 1)} \right) + 2 \text{Li}_2 \left( \frac{\lambda_1 - 2\lambda_2}{\lambda_1(\lambda_1 - \lambda_2 - 1)} \right) \\
- 2 \text{Li}_2 \left( \frac{\lambda_1 - \lambda_2 - 1}{\lambda_1 - 2\lambda_2} \right) - 2 \text{Li}_2 \left( \frac{2\lambda_2}{\lambda_1(\lambda_1 - \lambda_2 - 2)} \right) \\
+ 2 \text{Li}_2 \left( \frac{(\lambda_1 - \lambda_2 - 1)\lambda_2}{\lambda_1 - 2\lambda_2} \right) + 2 \text{Li}_2 \left( \frac{\lambda_2}{-\lambda_1 + \lambda_2 + 2} \right) - 2 \text{Li}_2 \left( \frac{1}{2} (-\lambda_1 + \lambda_2 + 2) \right) ,
\]

\[
\frac{\pi^2}{6} g_2(\lambda_1, \lambda_2)|_F = \lambda_2 - \frac{\pi^2}{2} - 1 + \log^2(2) \\
+ 2 \log(2) \log(\lambda_1 - \lambda_2 - 1) + \left( \frac{2}{\lambda_1} - 2\lambda_1 \right) \log(\lambda_1 - 1) + \left( \frac{\lambda_1}{2} - \frac{2}{\lambda_1} \right) \log(\lambda_1 - 2) \\
+ 2 \left( \lambda_1 - \lambda_2 + \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \log(\lambda_1 - \lambda_2) + \left( \lambda_2 - \frac{1}{\lambda_2} \right) \log(\lambda_1) \\
+ \log(\lambda_1 - 2\lambda_2) \left( -\frac{\lambda_1}{2} + \lambda_2 - \frac{1}{\lambda_2} + \frac{2}{\lambda_1} - 2 \log(2) \right) \\
+ 2 \log(\lambda_1) \log(\lambda_1 - \lambda_2) - \log(\lambda_1 - \lambda_2 - 1) - \log(\lambda_2) + \log(\lambda_1 - \lambda_2) \log(\lambda_2) \\
+ 2 \text{Li}_2 \left( \frac{1}{\lambda_1} \right) + 2 \text{Li}_2 (1 - \lambda_2) + 2 \text{Li}_2 \left( \frac{\lambda_1 - 2\lambda_2}{2(\lambda_1 - \lambda_2 - 1)} \right) - 2 \text{Li}_2 \left( \frac{\lambda_1 - 2\lambda_2}{\lambda_1(\lambda_1 - \lambda_2 - 1)} \right) \\
+ 2 \text{Li}_2 \left( \frac{\lambda_1 - \lambda_2 - 1}{\lambda_1 - 2\lambda_2} \right) + 2 \text{Li}_2 (\lambda_2) - 2 \text{Li}_2 \left( \frac{(\lambda_1 - \lambda_2 - 1)\lambda_2}{\lambda_1 - 2\lambda_2} \right) \\
+ 2 \text{Li}_2 \left( \frac{\lambda_2 - 1}{\lambda_2 - \lambda_1} \right) - 2 \text{Li}_2 \left( \frac{\lambda_1 - \lambda_1\lambda_2}{\lambda_1 - \lambda_2} \right) ,
\]

\[
\frac{\pi^2}{6} g_2(\lambda_1, \lambda_2)|_G = 0.
\]

References


