Chapter 9

Cohomology and Spectral Sequences

This appendix gives a short but intense introduction to cohomology and spectral sequences, two powerful but sometimes intimidating topics. It is included mainly for the convenience of the reader who is a non-specialist, so contains basic definitions and theorems (sometimes without proof) as well as illustrative examples. In particular, after reading the chapter, we hope the reader will feel competent to compute

- Higher derived functors (cohomology from a new viewpoint).
- Spectral sequences.

Comprehensive treatments of the material here (and proofs) may be found in Eisenbud [**11**], Hartshorne [**19**], and Weibel [**?**] Throughout this section, the ring R is a commutative, finitely generated $\mathbb{C}$–algebra, and the sheaf $\mathscr{O}_X$ is the sheaf of regular functions on a complex variety X.

0. Homological basics: Complexes and Resolutions

A sequence of R–modules and homomorphisms

$$\mathscr{C}: \cdots \xrightarrow{\phi_{j+2}} M_{j+1} \xrightarrow{\phi_{j+1}} M_j \xrightarrow{\phi_j} M_{j-1} \xrightarrow{\phi_{j-1}} \cdots$$

is a *complex* (or chain complex) if

$$\mathrm{im}\phi_{j+1} \subseteq \ker \phi_j.$$

The sequence is *exact* at M_j if $\mathrm{im}\phi_{j+1} = \ker \phi_j$; a complex which is exact everywhere is called an *exact sequence*. The j^{th} homology module of $\mathscr{C}$ is defined as:

$$H_j(\mathscr{C}) = \ker \phi_j / \mathrm{im}\phi_{j+1}.$$

The following complex is ubiquitous:

Example 0.1. [Koszul complex] Let $S = \mathbb{C}[x_1, \ldots, x_n]$, with $f_1, \ldots, f_m \in S$. Set $V = S^m$ with basis $\{e_1, \ldots, e_m\}$, and let $f = \sum_{i=1}^m f_i e_i$. The sequence:

$$0 \longrightarrow S \xrightarrow{d_n} \Lambda^1(V) \xrightarrow{d_{n-1}} \Lambda^2(V) \longrightarrow \cdots \xrightarrow{d_1} \Lambda^n(V) \cong S \longrightarrow 0 \ .$$

with $d_j(\gamma) = f \wedge \gamma$ is easily checked to be a complex; it is called the *Koszul complex* on $\{f_1, \ldots, f_m\}$. ◇

Example 0.2. sheaf theoretic version of something not exact as sequence of modules, but exact at sheaf level, in particular Toric with supp in B. ◇

0.1. Maps of complexes, Snake Lemma, long exact sequence.

Definition 0.3. If A and B are complexes, then a ***morphism of complexes*** ϕ is a family of homomorphisms $A_i \xrightarrow{\phi_i} B_i$ making the diagram below commute:

$$\begin{array}{lccccccccc}
A\colon & \cdots \longrightarrow & A_{i+1} & \xrightarrow{\partial_{i+1}} & A_i & \xrightarrow{\partial_i} & A_{i-1} & \xrightarrow{\partial_{i-1}} & \cdots \\
 & & \downarrow{\scriptstyle \phi_{i+1}} & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}} & & \\
B\colon & \cdots \longrightarrow & B_{i+1} & \xrightarrow{\delta_{i+1}} & B_i & \xrightarrow{\delta_i} & B_{i-1} & \xrightarrow{\delta_{i-1}} & \cdots
\end{array}$$

Lemma 0.4 (Induced Map on Homology)**.** *A morphism of complexes induces a map on homology.*

Proof. To show that ϕ_i induces a map $H_i(A) \to H_i(B)$, take $a_i \in A_i$ with $\partial_i(a_i) = 0$. Since the diagram commutes,

$$0 = \phi_{i-1}\partial_i(a_i) = \delta_i\phi_i(a_i)$$

Hence, $\phi_i(a_i)$ is in the kernel of δ_i, so we obtain a map $\ker \partial_i \to H_i(B)$. If $a_i = \partial_{i+1}(a_{i+1})$, then

$$\phi_i(a_i) = \phi_i\partial_{i+1}(a_{i+1}) = \delta_{i+1}\phi_{i+1}(a_{i+1}),$$

so ϕ takes the image of ∂ to the image of δ, yielding a map $H_i(A) \to H_i(B)$. □

When do two morphisms of complexes induce the same map on homology?

Definition 0.5. If A and B are complexes, and α, β are morphisms of complexes, then α and β are ***homotopic*** if there exists a family of homomorphisms $A_i \xrightarrow{\gamma_i} B_{i+1}$ such that for all i, $\alpha_i - \beta_i = \delta_{i+1}\gamma_i + \gamma_{i-1}\partial_i$. Notice that γ need not commute with ∂ and δ.

Theorem 0.6. *Homotopic maps induce the same map on homology.*

Proof. It suffices to show that if $\alpha_i = \delta_{i+1}\gamma_i + \gamma_{i-1}\partial_i$ then α induces the zero map on homology. But if $a_i \in H_i(A)$, then since $\partial_i(a_i) = 0$,

$$\alpha_i(a_i) = \delta_{i+1}\gamma_i(a_i) + \gamma_{i-1}\partial_i(a_i) = \delta_{i+1}\gamma_i(a_i) \in \mathrm{im}(\delta),$$

so $\alpha_i(a_i) = 0$ in $H_i(B)$. □

Lemma 0.7 (The Snake Lemma)**.** *For a commutative diagram of R–modules with exact rows*

$$\begin{array}{ccccccccc} & & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 & & \end{array}$$

then there exists an exact sequence:

$$\ker f_1 \longrightarrow \ker f_2 \longrightarrow \ker f_3 \xrightarrow{\delta} \mathrm{coker}\, f_1 \longrightarrow \mathrm{coker}\, f_2 \longrightarrow \mathrm{coker}\, f_3 \ .$$

Definition 0.8. A ***short exact sequence of complexes*** is a commuting diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ A : \cdots & \xrightarrow{\partial_3} & A_2 & \xrightarrow{\partial_2} & A_1 & \xrightarrow{\partial_2} & A_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ B : \cdots & \xrightarrow{\partial_3} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\partial_2} & B_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ C : \cdots & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_2} & C_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the columns are exact and the rows are complexes.

In Exercise 0.1 you'll prove the snake lemma, and in Exercise 0.2 you'll combine it with induction to show:

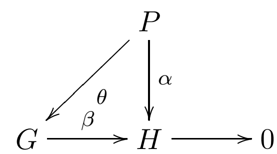
Theorem 0.9 (Long Exact Sequence in Homology)**.** *A short exact sequence of complexes yields a long exact sequence in homology:*

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

A proof using spectral sequences appears in §3.

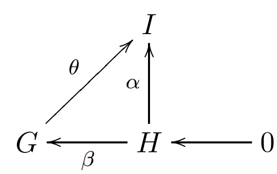
0.2. Projective and injective modules.

Definition 0.10. A module P is ***projective*** if it possesses a universal lifting property: For any R–modules G and H, given a homomorphism $P \xrightarrow{\alpha} H$ and surjection $G \xrightarrow{\beta} H$, there exists a homomorphism θ making the diagram below commute:



A (left) R-module M is *free* if M is isomorphic to a direct sum of copies of the (left) R–module R. Free modules are projective.

Definition 0.11. A module I is ***injective*** if given a homomorphism $H \xrightarrow{\alpha} I$ and injection $H \xrightarrow{\beta} G$, there exists a homomorphism θ making the diagram below commute:



Projective and Injective modules will come to the forefront in the next section, which describes *derived functors*.

0.3. Resolutions. Given an R–module M, there exists a projective module surjecting onto M; for example, take a free module with a generator for each element of M. This yields an exact sequence:

$$P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

The map d_0 has a kernel, so the process can be iterated, producing an exact sequence (possibly infinite) of free modules, terminating in M.

Definition 0.12. A ***projective resolution*** for an R–module M is an exact sequence of projective modules

$$\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0, \text{ with } \operatorname{coker}(d_1) = M.$$

Notice there is no uniqueness property; for example we could set $P_4' = P_4 \oplus R$ and $P_3' = P_3 \oplus R$, and define a map $P_4' \to P_3'$ which is the identity on the R–summands, and the original map on the P_i summands. In the category of R–modules the construction above shows that projective resolutions always exist. Surprisingly this is not the case for sheaves of $\mathscr{O}_X$-modules. In fact ([**19**], Exercise III.6.2), for $X = \mathbb{P}^1$ there is no projective object surjecting onto $\mathscr{O}_X$.

Definition 0.13. An ***injective resolution*** for an R–module M is an exact sequence of injective modules

$$I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2, \cdots \text{ with } \ker(d_0) = M.$$

While it is not obvious that injective resolutions exist, it can be shown (see, e.g. [**?**]) that in both the category of R–modules and the category of sheaves of $\mathscr{O}_X$–modules, every object does have an injective resolution.

Exercises for $\S$***0.***

0.1. Prove the snake lemma.

0.2. Prove the existance of the long exact sequence in homology.

0.3. Prove the existance of an injective resolution for R–modules, when $R = \mathbb{Z}$ (see Hungerford, divisibility section).

1. Functors and Derived Functors

In this section we describe the construction of derived functors, focusing on Ext^i (in the category of R-modules) and H^i (in the category of sheaves of $\mathscr{O}_X$–modules). For brevity we call these two categories “our categories”. Working in our categories keeps things concrete and lets us avoid introducing too much terminology, while highlighting the most salient features of the constructions, most of which apply in much more general contexts. For proofs and a detailed discussion, see [**11**] or [**?**].

1.1. Categories and Functors. Recall that a category is a class of objects, along with morphisms between the objects, satisfying certain properties: composition of morphisms is associative, and identity morphisms exist.

Definition 1.1. Suppose $\mathscr{B}$ and $\mathscr{C}$ are categories. A ***functor*** F is a function from $\mathscr{B}$ to $\mathscr{C}$, taking objects to objects and morphisms to morphisms, preserving identity morphisms and compositions. If

$$B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3$$

is a sequence of objects and morphisms in $\mathscr{B}$, then

- F is ***covariant*** if applying F yields a sequence of objects and morphisms in $\mathscr{C}$ of the form:

$$F(B_1) \xrightarrow{F(b_1)} F(B_2) \xrightarrow{F(b_2)} F(B_3).$$

- F is ***contravariant*** if applying F yields a sequence of objects and morphisms in $\mathscr{C}$ of the form:

$$F(B_3) \xrightarrow{F(b_2)} F(B_2) \xrightarrow{F(b_1)} F(B_1).$$

A functor is ***additive*** if it preserves addition of homomorphisms; this property will be necessary in the construction of derived functors.

Example 1.2. The global sections functor is covariant: given a sequence of $\mathscr{O}_X$–modules

$$\mathscr{M}_1 \xrightarrow{f} \mathscr{M}_2 \xrightarrow{g} \mathscr{M}_3,$$

taking global sections yields a sequence

$$\Gamma(\mathscr{M}_1) \to \Gamma(\mathscr{M}_2) \to \Gamma(\mathscr{M}_3).$$

◇

Definition 1.3. Let F be a functor from $\mathscr{B}$ to $\mathscr{C}$, with $\mathscr{B}$ and $\mathscr{C}$ categories of modules over a ring. Let

$$0 \longrightarrow B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3 \longrightarrow 0$$

be a short exact sequence. F is ***left–exact*** if either

- F is covariant, and the sequence
$$0 \longrightarrow F(B_1) \xrightarrow{F(b_1)} F(B_2) \xrightarrow{F(b_2)} F(B_3) \text{ is exact, or}$$
- F is contravariant, and the sequence
$$0 \longrightarrow F(B_3) \xrightarrow{F(b_2)} F(B_2) \xrightarrow{F(b_1)} F(B_1) \text{ is exact.}$$

A similar definition applies for right exactness; a functor F is said to be exact if it is both left and right exact, which is synonymous with saying that F preserves exact sequences.

1.2. Derived Functors. The construction of derived functors is motivated by the following question: if F is a left exact, contravariant functor and

$$0 \longrightarrow B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3 \longrightarrow 0$$

is a short exact sequence, then what is the cokernel of $F(b_1)$? For example, if N is a fixed R–module, then the functor $Hom_R(\bullet, N)$ is a functor of exactly this type.

Definition 1.4. Let $\mathscr{B}$ be the category of modules over a ring, and let F be a left exact, contravariant, additive functor from $\mathscr{B}$ to itself. If $M \in \mathscr{B}$, then there exists a projective resolution $P_\bullet$ for M.

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0$$

Applying F to $P_\bullet$ yields a complex:

$$0 \longrightarrow F(P_0) \xrightarrow{F(d_1)} F(P_1) \xrightarrow{F(d_2)} F(P_2) \longrightarrow \cdots .$$

The ***right derived functors*** $R^iF(M)$ are defined as

$$R^iF(M) = H^i(F(P_\bullet)).$$

Theorem 1.5. *$R^iF(M)$ is independent of the choice of projective resolution, and has the following properties:*

- $R^0F(M) = F(M)$.
- *If M is projective then $R^iF(M) = 0$ if $i > 0$.*
- *A short exact sequence*

  $$B_\bullet : 0 \to B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3 \to 0$$

  gives rise to a long exact sequence

  $$R^{j-1}F(B_3) \xrightarrow{R^{j-1}(F(b_2))} R^{j-1}F(B_2) \xrightarrow{R^{j-1}(F(b_1))} R^{j-1}F(B_1) \xrightarrow{\delta_{j-1}}$$
  $$R^{j}F(B_3) \xrightarrow{R^{j}(F(b_2))} R^{j}F(B_2) \xrightarrow{R^{j}(F(b_1))} R^{j}F(B_1) \xrightarrow{\delta_{j}}$$
  $$R^{j+1}F(B_3) \xrightarrow{R^{j+1}(F(b_2))} R^{j+1}F(B_2) \xrightarrow{R^{j+1}(F(b_1))} R^{j+1}F(B_1)$$

  of derived functors, where the connecting maps are natural: given another short exact sequence $C_\bullet$ and map from $B_\bullet$ to $C_\bullet$, the obvious diagram involving the R^iF commutes.

The proof is really not bad but long to write out, and we refer to Proposition A3.17 of [**11**] for details. There are four possible combinations of variance and exactness; the type of resolution used to compute the derived functors of F is given below:

F	*covariant*	*contravariant*
left exact	*injective*	*projective*
right exact	*projective*	*injective*

In the next sections, we study some common derived functors.

1.3. Ext. Let R be a ring, and suppose

$$B_\bullet : 0 \to B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3 \to 0$$

is a short exact sequence of R–modules, with N some fixed R–module. Applying $\mathrm{Hom}_R(\bullet, N)$ to $B_\bullet$ yields an exact sequence:

$$0 \to \mathrm{Hom}_R(B_3, N) \xrightarrow{c_1} \mathrm{Hom}_R(B_2, N) \xrightarrow{c_2} \mathrm{Hom}_R(B_1, N),$$

with $c_1(\phi) \mapsto \phi \circ b_2$ and $c_2(\theta) \mapsto \theta \circ b_1$; $\mathrm{Hom}_R(\bullet, N)$ is left exact and contravariant.

Definition 1.6. $\mathrm{Ext}^i_R(\bullet, N)$ is the i^{th} right derived functor of $\mathrm{Hom}_R(\bullet, N)$

Given R–modules M and N, to compute $\mathrm{Ext}^i_R(M, N)$, we must find a projective resolution for M

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0,$$

and compute the homology of the complex

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \mathrm{Hom}(P_2, N) \to \cdots$$

Example 1.7. Let $R = \mathbb{C}[x, y, z]$, $M = R/\langle xy, xz, yz\rangle$, and suppose $N \simeq R^1$. Applying $\mathrm{Hom}_R(\bullet, R^1)$ to the projective (indeed, free) resolution of M

$$0 \longrightarrow R(-3)^2 \xrightarrow{\begin{bmatrix} -z & -z \\ y & 0 \\ 0 & x \end{bmatrix}} R(-2)^3 \xrightarrow{\begin{bmatrix} xy & xz & yz \end{bmatrix}} R$$

simply means dualizing the module and transposing the differentials, so $\mathrm{Ext}^i(R/I, R)$ is:

$$H_i \left[0 \longrightarrow R \xrightarrow{\begin{bmatrix} xy \\ xz \\ yz \end{bmatrix}} R(2)^3 \xrightarrow{\begin{bmatrix} -z & y & 0 \\ -z & 0 & x \end{bmatrix}} R(3)^2 \longrightarrow 0 \right]$$

Thus, $\mathrm{Ext}^2(R/I, R)$ is the cokernel of the last map, and it is easy to check that $\mathrm{Ext}^0(R/I, R) = \mathrm{Ext}^1(R/I, R) = 0$. ◇

For a fixed R–module M, applying $\mathrm{Hom}_R(M, \bullet)$ to $B_\bullet$ yields an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M, B_1) \xrightarrow{c_1} \mathrm{Hom}_R(M, B_2) \xrightarrow{c_2} \mathrm{Hom}_R(M, B_3)\ ,$$

with $c_1(\phi) \mapsto b_1 \circ \phi$ and $c_2(\theta) \mapsto b_2 \circ \theta$; $\mathrm{Hom}_R(\cdot, M)$ is left exact and covariant. Thus, to compute the derived functors of $\mathrm{Hom}_R(\cdot, M)$, on a module N, we must find an injective resolution of N:

$$I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

then compute

$$H_i \left[0 \longrightarrow \mathrm{Hom}(I^0, M) \longrightarrow \mathrm{Hom}(I^1, M) \longrightarrow \mathrm{Hom}(I^2, M) \longrightarrow \cdots \right]$$

Using spectral sequences (see next section), it is possible to show that that $\mathrm{Ext}^i(M, N)$ can be regarded as the i^{th} derived functor of *either* $\mathrm{Hom}_R(\bullet, N)$ or $\mathrm{Hom}_R(M, \bullet)$.

1.4. The global sections functor. Let X be a variety, and suppose $\mathscr{B}$ is a coherent $\mathscr{O}_X$–module. As we saw in Chapter 7, the global sections functor Γ is left exact and covariant. Hence, to compute $L^i\Gamma(\mathscr{B})$, we take an injective resolution of $\mathscr{B}$:

$$\mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \mathscr{I}^2 \longrightarrow \cdots$$

then compute

$$H^i\left[\; 0 \longrightarrow \Gamma(\mathscr{I}^0) \longrightarrow \Gamma(\mathscr{I}^1) \longrightarrow \Gamma(\mathscr{I}^2) \longrightarrow \cdots \;\right]$$

In Example 1.7 we wrote down an explicit free resolution and computed the Ext–modules. Unfortunately, the general construction for injective resolutions produces very complicated objects. For example, if R is a polynomial ring, then the smallest injective module in which the residue field can be included is infinitely generated.

It is not obvious that there is a relation between the Čech cohomology which appeared in Chapter 7 and the derived functors of Γ defined above. At the end of this chapter, we'll see that there is a map

$$\check{H}^i(\mathscr{U}, \mathscr{F}) \longrightarrow H^i(X, \mathscr{F}),$$

and use spectral sequences to show that with certain conditions on $\mathscr{U}$ this is an isomorphism. The upshot is that many key facts about Cech cohomology (for example, the fact that a short exact sequence of sheaves gives rise to a long exact sequence in cohomology) follow automatically from the derived functor machinery!

1.5. Acyclic objects. The last concept we need in order to work with derived functors is the notion of an acyclic object.

Definition 1.8. Let F be a left–exact, covariant functor. An object A is ***acyclic*** for F if $R^iF(A) = 0$ for all $i > 0$. An acyclic resolution of M is an exact sequence

$$A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

where the A^i are acyclic, and $M = \ker(d^0)$.

The reason acyclic objects are important is that a resolution of acyclic objects is good enough to compute higher derived functors; in other words we have an alternative to using resolutions by projective or injective objects.

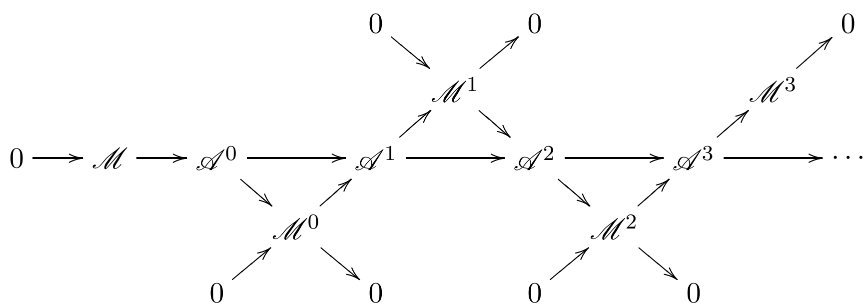
Theorem 1.9. *Let $\mathscr{M}$ be a coherent $\mathscr{O}_X$–module, and*

$$\mathscr{A}^0 \longrightarrow \mathscr{A}^1 \longrightarrow \mathscr{A}^2 \longrightarrow \cdots$$

a Γ–acyclic resolution of $\mathscr{M}$. Then

$$R^i\Gamma(\mathscr{M}) = H^i\left[\; 0 \longrightarrow \Gamma(\mathscr{A}^0) \longrightarrow \Gamma(\mathscr{A}^1) \longrightarrow \Gamma(\mathscr{A}^2) \longrightarrow \cdots \;\right]$$

Proof. First, break the resolution into short exact sequences:



Since the $\mathscr{A}^i$ are acyclic for Γ, applying Γ to the short exact sequence

$$0 \to \mathscr{M} \to \mathscr{A}^0 \to \mathscr{M}^0 \to 0$$

yields an exact sequence

$$0 \to \Gamma(\mathscr{M}) \to \Gamma(\mathscr{A}^0) \to \Gamma(\mathscr{M}^0) \to R^1\Gamma(\mathscr{M}) \to 0$$

Now apply the snake lemma to the middle two columns of the (exact, commutative) diagram below.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(\mathscr{M}) & \longrightarrow & \Gamma(\mathscr{A}^0) & \longrightarrow & \Gamma(\mathscr{M}^0) & \longrightarrow & R^1\Gamma(\mathscr{M}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(\mathscr{M}) & \longrightarrow & \Gamma(\mathscr{A}^0) & \longrightarrow & \Gamma(\mathscr{A}^1) & \longrightarrow & \Gamma(\mathscr{A}^1)/\Gamma(\mathscr{A}^0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & \longrightarrow & \Gamma(\mathscr{M}^1) & \longrightarrow & \Gamma(\mathscr{M}^1) & &
\end{array}$$

This yields a right exact sequence

$$0 \to R^1\Gamma(\mathscr{M}) \to \Gamma(\mathscr{A}^1)/\Gamma(\mathscr{A}^0) \to \Gamma(\mathscr{M}^1) \simeq \Gamma(\mathscr{A}^1)/\ker(d^1),$$

where $\Gamma(\mathscr{A}^1) \xrightarrow{d^1} \Gamma(\mathscr{A}^2)$. Hence,

$$R^1\Gamma(\mathscr{M}) = H^1\left[\, 0 \longrightarrow \Gamma(\mathscr{A}^0) \longrightarrow \Gamma(\mathscr{A}^1) \longrightarrow \Gamma(\mathscr{A}^2) \longrightarrow \cdots \,\right]$$

In Exercise 1.2 you'll show that iterating this process yields the theorem. □

Exercises for* $\S$*1.

1.1. Prove that the derived functors do not depend on choice of resolution. The key to this is to construct a homotopy between resolutions, and appeal to Theorem 1.6.

1.2. Complete the proof of Theorem 1.9 by replacing the sequence

$$0 \to \mathscr{M} \to \mathscr{A}^0 \to \mathscr{M}^0 \to 0$$

with

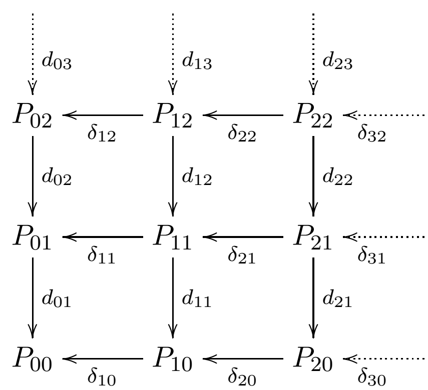
$$0 \to \mathscr{M}^{i-1} \to \mathscr{A}^i \to \mathscr{M}^i \to 0$$

2. Spectral Sequences

Spectral sequences are a fundamental tool in algebra and topology; at first glance, they can seem quite confusing. In this brief overview, we describe a specific type of spectral sequence, state the main theorem, and illustrate the use of spectral sequences by several examples.

2.1. Total complex of double complex.

Definition 2.1. A ***first quadrant double complex*** is a commuting diagram, where each row and each column is a complex:



For each antidiagonal, define a module

$$P_m = \bigoplus_{i+j=m} P_{ij}.$$

We may define maps

$$P_m \xrightarrow{D_m} P_{m-1}$$

via

$$D_m(c_{ij}) = d_{ij}(c_{ij}) + (-1)^m \delta_{ij}(c_{ij}).$$

Thus, $D_{m-1}D_m(a) = dd(a) + \delta\delta(a) \pm (d\delta(a) - \delta d(a))$. The fact that each row and each column are complexes implies that $\delta\delta(a) = 0$ and $dd(a) = 0$. The commutativity of the diagram implies that $d\delta(a) = \delta d(a)$, and so $D^2 = 0$.

Definition 2.2. The ***total complex*** $\mathrm{Tot}(P)$ associated to a double complex P_{ij} is the complex $(\mathscr{P}_\bullet, D_\bullet)$ defined above.

Definition 2.3. A ***filtration*** of a module M is a chain of submodules

$$0 \subseteq M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

A filtration has an associated graded object $\mathrm{gr}(M) = \oplus M_i/M_{i+1}$. The main theorem concerning the spectral sequence of a double complex describes two different filtrations of the homology of the associated single complex. To describe these filtrations, we need to follow two different paths through the double complex.

2.2. The vertical filtration. For a double complex as above, we first compute homology with respect to the vertical differentials, yielding the following diagram:

$$\begin{array}{ccccccc}
\ker(d_{02})/\mathrm{im}(d_{03}) & \xleftarrow[\delta_{12}]{} & \ker(d_{12})/\mathrm{im}(d_{13}) & \xleftarrow[\delta_{22}]{} & \ker(d_{22})/\mathrm{im}(d_{23}) & \leftarrow\cdots \\
\ker(d_{01})/\mathrm{im}(d_{02}) & \xleftarrow[\delta_{11}]{} & \ker(d_{11})/\mathrm{im}(d_{12}) & \xleftarrow[\delta_{21}]{} & \ker(d_{21})/\mathrm{im}(d_{22}) & \leftarrow\cdots \\
P_{00}/\mathrm{im}(d_{01}) & \xleftarrow[\delta_{10}]{} & P_{10}/\mathrm{im}(d_{11}) & \xleftarrow[\delta_{20}]{} & P_{20}/\mathrm{im}(d_{21}) & \leftarrow\cdots
\end{array}$$

These objects are renamed as follows:

$$\begin{array}{ccccccc}
{}_{\mathrm{vert}}E^1_{02} & \xleftarrow[\delta_{12}]{} & {}_{\mathrm{vert}}E^1_{12} & \xleftarrow[\delta_{22}]{} & {}_{\mathrm{vert}}E^1_{22} & \xleftarrow[\delta_{32}]{}\cdots \\
{}_{\mathrm{vert}}E^1_{01} & \xleftarrow[\delta_{11}]{} & {}_{\mathrm{vert}}E^1_{11} & \xleftarrow[\delta_{21}]{} & {}_{\mathrm{vert}}E^1_{21} & \xleftarrow[\delta_{31}]{}\cdots \\
{}_{\mathrm{vert}}E^1_{00} & \xleftarrow[\delta_{10}]{} & {}_{\mathrm{vert}}E^1_{10} & \xleftarrow[\delta_{20}]{} & {}_{\mathrm{vert}}E^1_{20} & \xleftarrow[\delta_{30}]{}\cdots
\end{array}$$

The vertical arrows disappeared after computing homology with respect to d, and the horizontal arrows reflect the induced maps on homology from the original diagram. Now, compute the homology of the diagram above, with respect to the horizontal maps. For example, the object ${}_{\mathrm{vert}}E^2_{11}$ represents

$$\ker({}_{\mathrm{vert}}E^1_{11} \xrightarrow{\delta_{11}} {}_{\mathrm{vert}}E^1_{01})/\mathrm{im}({}_{\mathrm{vert}}E^1_{21} \xrightarrow{\delta_{21}} {}_{\mathrm{vert}}E^1_{11})$$

The resulting modules may be displayed in a grid:

$$
\begin{array}{ccc}
{}_{\text{vert}}E^2_{02} & {}_{\text{vert}}E^2_{12} & {}_{\text{vert}}E^2_{22} \\
\\
{}_{\text{vert}}E^2_{01} & {}_{\text{vert}}E^2_{11} & {}_{\text{vert}}E^2_{21} \\
\\
{}_{\text{vert}}E^2_{00} & {}_{\text{vert}}E^2_{10} & {}_{\text{vert}}E^2_{20}
\end{array}
$$

Although it appears at first that there are no maps between these objects, the crucial observation is that there is a map $d^2_{i,j}$ from E^2_{ij} to $E^2_{i-2,j+1}$. This "knight's move" is constructed just like the connecting map δ appearing in the snake lemma. The diagram above (with differentials added) is thus:

$$
\begin{array}{ccc}
{}_{\text{vert}}E^2_{02} & {}_{\text{vert}}E^2_{12} & {}_{\text{vert}}E^2_{22} \\
 & \nwarrow d^2_{21} & \\
{}_{\text{vert}}E^2_{01} & {}_{\text{vert}}E^2_{11} & {}_{\text{vert}}E^2_{21} \\
 & \nwarrow d^2_{20} & \\
{}_{\text{vert}}E^2_{00} & {}_{\text{vert}}E^2_{10} & {}_{\text{vert}}E^2_{20}
\end{array}
$$

So we may compute homology with respect to this differential. The homology at position (i,j) is labeled, as one might expect, ${}_{\text{vert}}E^3_{ij}$; it is now the case (but far from intuitive) that there is a differential $d^3_{i,j}$ taking ${}_{\text{vert}}E^3_{ij}$ to ${}_{\text{vert}}E^3_{i-3,j+2}$:

$$
\begin{array}{cccc}
{}_{\text{vert}}E^3_{02} & {}_{\text{vert}}E^3_{12} & {}_{\text{vert}}E^3_{22} & {}_{\text{vert}}E^3_{32} \\
\\
{}_{\text{vert}}E^3_{01} & {}_{\text{vert}}E^3_{11} \; \nwarrow d^3_{30} & {}_{\text{vert}}E^3_{21} & {}_{\text{vert}}E^3_{31} \\
\\
{}_{\text{vert}}E^3_{00} & {}_{\text{vert}}E^3_{10} & {}_{\text{vert}}E^3_{20} & {}_{\text{vert}}E^3_{30}
\end{array}
$$

The process continues, with d^r_{ij} mapping ${}_{\text{vert}}E^r_{ij}$ to ${}_{\text{vert}}E^r_{i-r,j+r-1}$. One thing that is obvious is that since the double complex lies in the first quadrant, eventually the differentials in and out at position (i,j) must be zero, so that the module at position (i,j) stabilizes; it is written ${}_{\text{vert}}E^\infty_{ij}$. For example, it is easy to see that ${}_{\text{vert}}E^2_{10} = {}_{\text{vert}}E^\infty_{10}$, while ${}_{\text{vert}}E^2_{20} \neq {}_{\text{vert}}E^\infty_{20}$ but ${}_{\text{vert}}E^3_{20} = {}_{\text{vert}}E^\infty_{20}$.

2.3. Main theorem. The main theorem is that the E^∞ terms of a spectral sequence from a first quadrant double complex are related to the homology of the total complex.

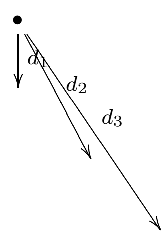
Definition 2.4. If $gr(M)_m \simeq \bigoplus_{i+j=m} E_{ij}^\infty$, then we say that a spectral sequence of the filtered object M ***converges***, and write

$$E^r \Rightarrow M$$

Theorem 2.5. *For the filtration of $H_m(\mathrm{Tot})$ obtained by truncating columns of the double complex,*

$$\bigoplus_{i+j=m} {}_{\mathrm{vert}} E_{ij}^\infty \Rightarrow H_m(\mathrm{Tot}).$$

As with the long exact sequence of of derived functors, the proof is not bad, but lengthy, so we refer to [**?**] or [**11**] for details. In the previous section, we first computed homology with respect to the vertical differential d. If instead we first compute homology with respect to the horizontal differential δ, then the higher differentials are:



As before, for $r \gg 0$, the source and target are zero, so the homology at position (i, j) stabilizes. The resulting value is denoted ${}_{\mathrm{hor}} E_{ij}^\infty$, and we have:

Theorem 2.6. *For the filtration of $H_m(\mathrm{Tot})$ obtained by truncating rows of the double complex,*

$$\bigoplus_{i+j=m} {}_{\mathrm{hor}} E_{ij}^\infty \Rightarrow H_m(\mathrm{Tot}).$$

For a first quadrant double complex (the only kind that will be of interest to us), the above two theorems tell us that

$$\bigoplus_{i+j=m} {}_{\mathrm{hor}} E_{ij}^\infty \Rightarrow H_m(\mathrm{Tot}) \text{ and } \bigoplus_{i+j=m} {}_{\mathrm{vert}} E_{ij}^\infty \Rightarrow H_m(\mathrm{Tot}).$$

Because the filtrations for the horizontal and vertical spectral sequence are different, it is often the case that for one of the spectral sequences the E^∞ terms stabilize very early (perhaps even vanishing). So the main idea is to play off the two different filtrations against each other. This is illustrated in the next example.

Example 2.7. We prove Theorem 0.9 via spectral sequences. Let $0 \to C_2 \to C_1 \to C_0 \to 0$ be a short exact sequence of complexes:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
C_2 : 0 & \leftarrow & C_{02} & \leftarrow & C_{12} & \leftarrow & C_{22} & \leftarrow\cdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
C_1 : 0 & \leftarrow & C_{01} & \leftarrow & C_{11} & \leftarrow & C_{21} & \leftarrow\cdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
C_0 : 0 & \leftarrow & C_{00} & \leftarrow & C_{10} & \leftarrow & C_{20} & \leftarrow\cdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

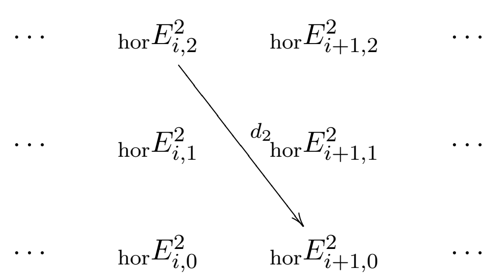
Since the columns are exact, it is immediate that for all (i, j)

$$_{\text{vert}}\, E^1_{ij} = {}_{\text{vert}}\, E^\infty_{ij} = 0$$

By Theorem 2.5, we conclude $H_m(\text{Tot}) = 0$ for all m. For the horizontal filtration $_{\text{hor}}\, E^1_{ij} = H_i(C_j)$ if $j \in \{1, 2, 3\}$, and 0 otherwise. For E^2

$$_{\text{hor}}\, E^2_{ij} = \begin{cases} \ker(H_i(C_1) \to H_i(C_2)) & j = 1 \\ \ker(H_i(C_2) \to H_i(C_3))/\text{im}(H_i(C_1) \to H_i(C_2)) & j = 2 \\ \text{coker}(H_i(C_2) \to H_i(C_3)) & j = 3. \end{cases}$$

The d_2 differential is zero for the middle row, and maps $_{\text{hor}}E^2_{i,2} \to {}_{\text{hor}}E^2_{i+1,0}$:



So $_{\text{hor}}E^2_{i,1} = {}_{\text{hor}}E^\infty_{i,1}$, while

$$_{\text{hor}}E^3_{i,2} = {}_{\text{hor}}E^\infty_{i,2} = \ker({}_{\text{hor}}E^2_{i,2} \to {}_{\text{hor}}E^2_{i+1,0})$$

and

$$_{\text{hor}}E^3_{i,0} = {}_{\text{hor}}E^\infty_{i,0} = \text{coker}({}_{\text{hor}}E^2_{i,2} \to {}_{\text{hor}}E^2_{i+1,0})$$

By Theorem 2.6,

$$H_m(\text{Tot}) = \bigoplus_{i+j=m} {}_{\text{hor}}\, E^\infty_{ij}$$

From the vertical spectral sequence, $H_m(\mathrm{Tot}) = 0$, so all the terms ${}_{\mathrm{hor}}E^{\infty}_{ij}$ must vanish. Working backwards, we see this means

$$0 = {}_{\mathrm{hor}}E^2_{i,1} = \ker(H_i(C_2) \to H_i(C_3))/\mathrm{im}(H_i(C_1) \to H_i(C_2)),$$

hence $H_i(C_1) \to H_i(C_2) \to H_i(C_3)$ is exact, and

$$\ker(H_i(C_1) \to H_i(C_2)) \simeq \mathrm{coker}(H_{i+1}(C_2) \to H_{i+1}(C_3))$$

which yields the long exact sequence in homology. $\diamond$

Exercises for* $\S$*2.

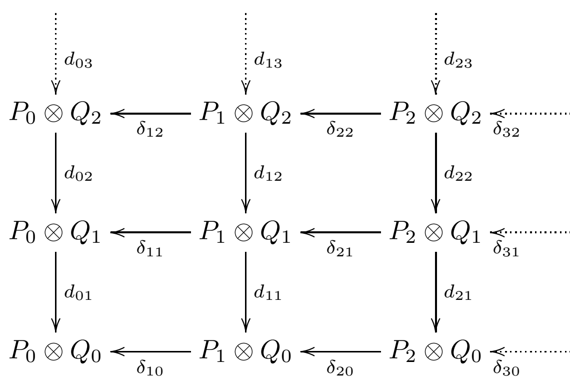
2.1. Tensor product is right exact and covariant. Prove that the i^{th} left derived functor of $\bullet \otimes_R N$ is isomorphic to the i^{th} left derived functor of $N \otimes_R \bullet$ as follows: Let

$$\cdots \to P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0$$

be a projective resolution for M and

$$\cdots \to Q_2 \xrightarrow{q_2} Q_1 \xrightarrow{q_1} Q_0$$

be a projective resolution for N. Form the double complex



with differentials $P_i \otimes Q_j \xrightarrow{\delta_{ij}} P_{i-1} \otimes Q_j$ defined by $a \otimes b \mapsto p_i(a) \otimes b$, and $P_i \otimes Q_j \xrightarrow{d_{ij}} P_i \otimes Q_{j-1}$ defined by $a \otimes b \mapsto a \otimes q_j(b)$.

(a) Show that for the vertical filtration, the E^1 terms are

$${}_{vert}E^1_{ij} = \begin{cases} P_i \otimes N & j = 0 \\ 0 & j \neq 0. \end{cases}$$

Now explain why ${}_{vert}E^2 = {}_{vert}E^{\infty}$, and these terms are:

$${}_{vert}E^2_{ij} = \begin{cases} H_i(P_\bullet \otimes N) = Tor_i(M, N) & j = 0 \\ 0 & j \neq 0. \end{cases}$$

(b) Show that for the horizontal filtration

$${}_{hor}E^2_{ij} = \begin{cases} H_j(M \otimes Q_\bullet) = Tor_j(N, M) & i = 0 \\ 0 & i \neq 0. \end{cases}$$

(c) Put everything together to conclude that

$$Tor_m(M,N) = \bigoplus_{i+j=m} {}_{\text{vert}}E_{ij}^{\infty} \simeq gr(H_m(\text{Tot})) \simeq \bigoplus_{i+j=m} {}_{\text{hor}}E_{ij}^{\infty} \simeq Tor_m(N,M)$$

2.2. Prove that $\text{Ext}^i(M,N)$ can be regarded as the i^{th} derived functor of *either* $\text{Hom}_R(\bullet, N)$ or $\text{Hom}_R(M, \bullet)$. The method is quite similar to the proof above, except for this one, you'll need both projective and injective resolutions.

3. Spectral Sequences and Derived Functors

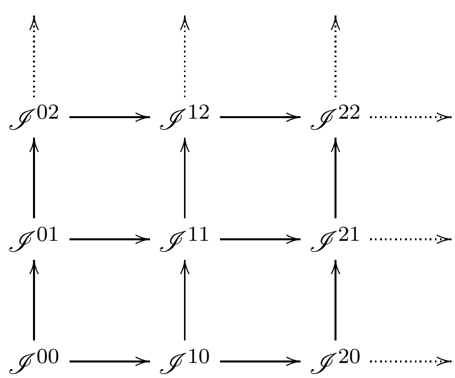
In this last section, we'll see how useful the machinery of spectral sequences in yielding theorems about derived functors. To do this, we first define resolutions of complexes. Note that sometimes our differentials on the double complex go "up and right" instead of "down and left", so the higher differentials change accordingly.

3.1. Resolution of a complex.

Suppose

$$A : 0 \longrightarrow \mathscr{A}^0 \longrightarrow \mathscr{A}^1 \longrightarrow \mathscr{A}^2 \longrightarrow \cdots$$

is a complex, either of R–modules or of sheaves of $\mathscr{O}_X$–modules. An *injective resolution* of A is a double complex:



satisfying the following properties (d^{jk} denotes the horizontal differential at (j,k)).

- The complex is exact.
- Each column $\mathscr{I}^{i*}$ is an injective resolution of $\mathscr{A}^i$.
- $\ker(d^{jk})$ is an injective summand of $\mathscr{I}^{jk}$.

The last condition implies that $\mathrm{im}(d^{j,k})$ is also injective. This yields a "Hodge decomposition":

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{im}(d^{j-1,k}) & \longrightarrow & \ker(d^{j,k}) & \longrightarrow & H_d^{j,k} & \longrightarrow & 0 \\ & & \downarrow & \swarrow & & & & & \\ & & \mathrm{im}(d^{j-1,k}) & & & & & & \end{array}$$

It follows that we may decompose the sequence

$$\mathscr{I}^{j-1,k} \xrightarrow{d^{j-1,k}} \mathscr{I}^{j,k} \xrightarrow{d^{j,k}} \mathscr{I}^{j+1,k}$$

as:

$$\begin{array}{ccccccc} \cdots\to \mathrm{im}(d^{j-2,k}) & \xrightarrow{0} & \mathrm{im}(d^{j,k}) & \xrightarrow{1} & \mathrm{im}(d^{j,k}) \cdots\to \\ \bigoplus & & \bigoplus & & \bigoplus \\ \cdots\to H^{j-1,k} & \xrightarrow{0} & H^{j,k} & \xrightarrow{0} & H^{j+1,k} \cdots\to \\ \bigoplus & & \bigoplus & & \bigoplus \\ \cdots\to \mathrm{im}(d^{j-1,k}) & \xrightarrow{1} & \mathrm{im}(d^{j-1,k}) & \xrightarrow{0} & \mathrm{im}(d^{j+1,k}) \cdots\to \end{array}$$

An inductive argument (Exercise 3.1) shows that in a category with enough injective objects, injective resolutions of complexes always exist.

3.2. Grothendieck spectral sequence. One of the most important spectral sequences is due to Grothendieck, and relates the higher derived functors of a pair of functors F,G, and their composition FG.

Theorem 3.1. *Suppose that F is a left exact, covariant functor from $\mathscr{C}_1 \to \mathscr{C}_2$ and G is a left exact, covariant functor from $\mathscr{C}_2 \to \mathscr{C}_3$, where the $\mathscr{C}_i$ are one of our categories. If $A \in \mathscr{C}_1$ has an F–acyclic resolution $\mathscr{A}^\bullet$ such that $F(\mathscr{A}^i)$ is G–acyclic then*

$$R^iG(R^jF(A)) \Rightarrow R^{i+j}GF(A)$$

Proof. Take an injective resolution $\mathscr{I}^{\bullet,\bullet}$ for the complex

$$0 \longrightarrow F(\mathscr{A}^0) \longrightarrow F(\mathscr{A}^1) \longrightarrow F(\mathscr{A}^2) \longrightarrow \cdots$$

Apply G to $\mathscr{I}^{\bullet,\bullet}$. It follows from the construction above that a row of the double complex $G(\mathscr{I}^{\bullet,\bullet})$ has the form:

$$\begin{array}{ccccccc}
\cdots\to & G(\mathrm{im}(d^{j-2,k})) & \xrightarrow{G(0)} & G(\mathrm{im}(d^{j,k})) & \xrightarrow{G(1)} & G(\mathrm{im}(d^{j,k})) & \cdots\to \\
& \bigoplus & & \bigoplus & & \bigoplus & \\
\cdots\to & G(H^{j-1,k}) & \xrightarrow{G(0)} & G(H^{j,k}) & \xrightarrow{G(0)} & G(H^{j+1,k}) & \cdots\to \\
& \bigoplus & & \bigoplus & & \bigoplus & \\
\cdots\to & G(\mathrm{im}(d^{j-1,k})) & \xrightarrow{G(1)} & G(\mathrm{im}(d^{j-1,k})) & \xrightarrow{G(0)} & G(\mathrm{im}(d^{j+1,k})) & \cdots\to
\end{array}$$

Hence,

$$_{hor}E^1_{ij} = G(H^{i,j})$$

By construction, $H^{i,j}$ is the j^{th} object in an injective resolution for the i^{th} cohomology of $F(\mathscr{A}^\bullet)$. Since $\mathscr{A}^\bullet$ was an F–acyclic resolution for A, the i^{th} cohomology is exactly $R^iF(A)$, so that

$$_{hor}E^2_{ij} = H^j\left[0 \to G(H^{i,0}) \to G(H^{i,1}) \to G(H^{i,2}) \to \cdots\right] = R^jG(R^iF(A))$$

Next, we turn to the vertical filtration. We have the double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
G(\mathscr{I}^{02}) & \longrightarrow & G(\mathscr{I}^{12}) & \longrightarrow & G(\mathscr{I}^{22}) & \cdots\to & \\
\uparrow & & \uparrow & & \uparrow & & \\
G(\mathscr{I}^{01}) & \longrightarrow & G(\mathscr{I}^{11}) & \longrightarrow & G(\mathscr{I}^{21}) & \cdots\to & \\
\uparrow & & \uparrow & & \uparrow & & \\
G(\mathscr{I}^{00}) & \longrightarrow & G(\mathscr{I}^{10}) & \longrightarrow & G(\mathscr{I}^{20}) & \cdots\to &
\end{array}$$

Since $\mathscr{I}^{ij}$ is an injective resolution of $F(\mathscr{A}^i)$,

$$R^jG(F(\mathscr{A}^i)) = H^j\left[0 \to G(\mathscr{I}^{i0}) \to G(\mathscr{I}^{i1}) \to G(\mathscr{I}^{i2}) \to \cdots\right].$$

Now, the assumption that the $F(\mathscr{A}^i)$ are G–acyclic forces $R^jG(F(\mathscr{A}^i))$ to vanish, for all $j > 0$! Hence, the cohomology of a *column* of the double complex above vanishes, except at position zero. In short

$$_{vert}E^1_{ij} = \begin{cases} GF(\mathscr{A}^i) & j = 0 \\ 0 & j \neq 0. \end{cases}$$

Thus,

$$ {}_{vert}E^{\infty}_{ij} = {}_{vert}E^2_{ij} = \begin{cases} R^{i+j}GF(A) & j = 0 \\ 0 & j \neq 0. \end{cases} $$

Applying Theorem 2.5 and Theorem 2.6 concludes the proof. □

3.3. Comparing cohomology theories. Our final application of spectral sequences will be to relate the higher derived functors of Γ to the Čech cohomology. For any map $Y \overset{f}{\hookrightarrow} X$ and sheaf $\mathscr{F}$ on Y, the pushforward is defined via:

$$ f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V)) $$

If I_p denotes a $p+1$-tuple $\{i_0 < i_1 < \cdots < i_p\}$ and $U_{I_p} = U_{i_0} \cap \cdots \cap U_{i_p}$, then applying this to the inclusion $U_{I_p} \overset{i}{\hookrightarrow} X$ gives a sheaf theoretic version of the Čech complex.

$$ \mathscr{C}^p(\mathscr{U}, \mathscr{F}) = \prod_{I_p} i_*(\mathscr{F}|_{U_{I_p}}). $$

In Exercise 3.2 you'll show that $\mathscr{C}^\bullet$ gives a resolution of $\mathscr{F}$, and if $\mathscr{F}$ is injective, so are the $\mathscr{C}^\bullet$. Taking this as given, we then have:

Lemma 3.2. *For an open cover $\mathscr{U}$, there is a map*

$$ \check{H}^i(\mathscr{U}, \mathscr{F}) \longrightarrow H^i(X, \mathscr{F}) $$

Proof. Take an injective resolution $\mathscr{I}^\bullet$ for $\mathscr{F}$. By injectivity, we get

$$ \begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{C}^0 & \longrightarrow & \mathscr{C}^1 & \longrightarrow & \cdots \\ & & \downarrow & \swarrow & & & & & \\ & & \mathscr{I}^0 & & & & & & \end{array} $$

Iterating the construction gives a map of complexes $\mathscr{C}^\bullet \to \mathscr{I}^\bullet$, which by Lemma 0.4 yields a map on cohomology. □

Theorem 3.3. *Let $\mathscr{U}$ be an open cover such that for any I_p,*

$$ H^i(U_{I_p}, \mathscr{F}) = 0, \textit{ for all } i \geq 1. $$

Then

$$ \check{H}^i(\mathscr{U}, \mathscr{F}) = H^i(X, \mathscr{F}). $$

Proof. Take an injective resolution $\mathscr{I}^\bullet$ for $\mathscr{F}$. The hypothesis that $H^i(U_{I_p}, \mathscr{F}) = 0, i > 0$ implies that the sequence

$$ 0 \longrightarrow \mathscr{F}(U_{I_p}) \longrightarrow \mathscr{I}^0(U_{I_p}) \longrightarrow \mathscr{I}^1(U_{I_p}) \longrightarrow \cdots $$

is exact. Then as in the construction of the sheaf-theoretic Čech complex, we obtain a Čech complex built out of the direct product of these, which is by construction a resolution (depicted below) of the Čech complex for $\mathscr{F}$. The bottom row is included for clarity, it is *not* part of the complex.

$$
\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
\mathscr{C}^0(\mathscr{U},\mathscr{I}^2) & \longrightarrow & \mathscr{C}^1(\mathscr{U},\mathscr{I}^2) & \longrightarrow & \mathscr{C}^2(\mathscr{U},\mathscr{I}^2) & \cdots\cdots> & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathscr{C}^0(\mathscr{U},\mathscr{I}^1) & \longrightarrow & \mathscr{C}^1(\mathscr{U},\mathscr{I}^1) & \longrightarrow & \mathscr{C}^2(\mathscr{U},\mathscr{I}^1) & \cdots\cdots> & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathscr{C}^0(\mathscr{U},\mathscr{I}^0) & \longrightarrow & \mathscr{C}^1(\mathscr{U},\mathscr{I}^0) & \longrightarrow & \mathscr{C}^2(\mathscr{U},\mathscr{I}^0) & \cdots\cdots> & \\
\vdots & & \vdots & & \vdots & & \\
\mathscr{C}^0(\mathscr{U},\mathscr{F}) & \longrightarrow & \mathscr{C}^1(\mathscr{U},\mathscr{F}) & \longrightarrow & \mathscr{C}^2(\mathscr{U},\mathscr{F}) & \cdots\cdots> &
\end{array}
$$

Applying Γ, since $H^i(U_{I_p}, \mathscr{F}) = 0$ for $i > 0$,

$$ {}_{vert}E^1_{ij} = \begin{cases} \Gamma(\mathscr{C}^i(\mathscr{U},\mathscr{F})) & j = 0 \\ 0 & j \neq 0. \end{cases} $$

Thus, $E^2 = E^\infty$, and since $\Gamma(\mathscr{C}^i(\mathscr{U},\mathscr{F})) = C^i(\mathscr{U},\mathscr{F})$

$$ {}_{vert}E^2_{ij} = \begin{cases} \check{H}^i(\mathscr{U},\mathscr{F}) & j = 0 \\ 0 & j \neq 0. \end{cases} $$

For the horizontal filtration, since the $\mathscr{C}^i(\mathscr{U},\mathscr{I}^j)$ are injective,

$$ {}_{hor}E^1_{ij} = \begin{cases} \Gamma(\mathscr{I}^j) & i = 0 \\ 0 & i \neq 0. \end{cases} $$

and thus

$$ {}_{hor}E^2_{ij} = \begin{cases} H^j(\Gamma(\mathscr{I}^\bullet)) & i = 0 \\ 0 & i \neq 0. \end{cases} $$

But this is the usual derived functor cohomology. The result follows from Theorem 2.5 and Theorem 2.6. □

Exercises for* $\S$*3.

3.1. Prove that in a category with enough injective objects, injective resolutions of complexes always exist.

3.2. Show that $\mathscr{C}^\bullet$ is a resolution of $\mathscr{F}$, as follows. By working at the level of stalks, show that there is a morphism of complexes

$$ \mathscr{C}^i(\mathscr{U},\mathscr{F})_p \xrightarrow{k} \mathscr{C}^{i-1}(\mathscr{U},\mathscr{F})_p $$

such that $(d_{i-1}k + kd_i)$ is the identity. Conclude by applying Theorem 0.6. Finally, show that if $\mathscr{F}$ is injective, then so are the sheaves $\mathscr{C}^i(\mathscr{U},\mathscr{F})$. If you get stuck, see [**19**], III.4.

3.3. Let $Y \overset{f}{\hookrightarrow} X$ be a continuous map between topological spaces, with A a sheaf of abelian groups on Y.

(a) Show that pushforward f_* is left exact and covariant, so associated to A are objects $R^j f_*(A))$.

(b) Use Theorem 3.1 to obtain the *Leray spectral sequence*:

$$H^i(R^j f_*(A)) \Rightarrow H^{i+j}(A)$$

3.4. Suppose $\mathscr{U}$ is a *Leray* cover: $\check{H}^i(U, \mathscr{F}) = 0$ for all open sets $U \in \mathscr{U}$. Show that then $\check{H}^i(\mathscr{U}, \mathscr{F}) = \check{H}^i(X, \mathscr{F})$.