On a conjecture of Rose

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Abstract

Let Δ be a triangulation of a topological n ball embedded in $\mathbf{R}^n$. Rose has conjectured that if the module $C^r(\hat{\Delta})$ of piecewise polynomial functions of smoothness r on $\hat{\Delta}$ is free, then so is $C^{r-1}(\hat{\Delta})$. For $n = 2$, we prove the conjecture, and we show that it is false in all higher dimensions.

1 Introduction

Let Δ be a finite simplicial complex which triangulates a n-ball, embedded in $\mathbf{R}^n$. A fundamental question in approximation theory is the problem of determining the dimension of the vector space $C^r_k(\Delta)$ of piecewise polynomial functions on Δ, of smoothness r and degree at most k. In the planar ($n = 2$) case, Alfeld and Schumaker [1] give a formula for dim $C^r_k(\Delta), k \geq 3r + 1$. By embedding $\mathbf{R}^n$ in $\mathbf{R}^{n+1}$ and forming the cone $\hat{\Delta}$ of Δ with the origin in $\mathbf{R}^{n+1}$, the problem becomes a question about graded modules. In a series of papers ([2],[3],[4]) Billera and Rose pioneered the use of algebraic methods in studying the problem, an approach further developed by Schenck and Stillman ([5], [6],[7]). A particularly nice situation occurs when the module $C^r(\hat{\Delta})$ is free, for then the dimension of $C^r_k(\Delta)$ is determined solely by local data (see [5]), i.e. the global geometry plays no role. Rose has conjectured that for a fixed Δ, $C^r(\hat{\Delta})$ free implies $C^{r-1}(\hat{\Delta})$ free. We prove that this is true for $n = 2$, but false in general.

2 A proof

Let $R = \mathbf{R}[x_0, \ldots, x_n]$, and let $r \in \mathbf{Z}_{\geq 0}$. Δ_i, Δ^0_i will denote (respectively) the sets of i-dimensional faces and i-dimensional interior faces of Δ, and f_i and f^0_i denote the number of elements in these sets. $\hat{\Delta}$ will denote the join of Δ (embedded in the hyperplane $\{x_0 = 1\} \subseteq \mathbf{R}^{n+1}$) with the origin in $\mathbf{R}^{n+1}$. In the remainder of this section, let $n = 2$. For $\epsilon \in \Delta^0_1$, let l_ϵ be a nonzero homogeneous linear form vanishing on $\hat{\epsilon}$, and for $v_i \in \Delta^0_0$, define $J_r(v_i) = \sum_{v_i \in \epsilon} < l_\epsilon^{r+1} >$, i.e. $J_r(v_i)$ is the ideal generated by the $r + 1^{st}$ powers of the (homogenizations of the) linear forms vanishing on edges terminating in v_i.

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In [6], we showed that in the $n = 2$ case, a certain local cohomology module $H^0_m(M_r)$ determines if $C^r(\hat{\Delta})$ is free; that module also measures the deviation of the actual spline space dimension from value given by the Alfeld-Schumaker formula. We cite the relevant results here, and refer the reader to [6] for more details.

Lemma 2.1 *(Schenck and Stillman, [6]) Let Δ be a disk in the plane, and define $K^r \subset \bigoplus_{\epsilon_i \in \Delta_1^0} R\epsilon_i$ to be the submodule generated by*

$$\{\epsilon_i \mid \epsilon_i \textit{ not totally interior}\},$$

(an edge is totally interior if both its vertices are interior), and for each interior vertex v the set

$$\{\sum_{v \in \epsilon_i} a_{\epsilon_i}\epsilon_i \mid \textit{there exists a relation } \sum_{v\in\epsilon_i} a_{\epsilon_i} l_{\epsilon_i}^{r+1} = 0, \textit{for } a_{\epsilon_i} \in R\}.$$

Then $H^0_m(M_r)$ is given by generators and relations by:

$$0 \longrightarrow K^r \longrightarrow \bigoplus_{\epsilon_i \in \Delta_1^0} R\epsilon_i \longrightarrow H^0_m(M_r) \longrightarrow 0.$$

Lemma 2.2 *(Schenck and Stillman, [6]) If $r \in \mathbf{Z}_{\geq 0}$, then the following are equivalent:*
(a) $C^r(\hat{\Delta})$ is a free R-module;
(b) $H^0_m(M_r)$ vanishes;
(c) $\dim C^r_k(\Delta)$ is given by the Alfeld-Schumaker formula, for all k.

In the plane, Rose's conjecture is true:

Theorem 2.3 *Let Δ be a topological disk. If $C^r(\hat{\Delta})$ is a free module, then $C^{r-1}(\hat{\Delta})$ is also free.*

Proof. By Lemma 2.2, it suffices to show:

$$H^0_m(M_r) = 0 \longrightarrow H^0_m(M_{r-1}) = 0.$$

From the presentation of Lemma 2.1, $H^0_m(M_r)$ is generated in a single degree; we are assuming that it is zero, so it must be annihilated by degree zero syzygies. Since all syzygies are homogeneous, the degree zero syzygies are generated by local degree zero syzygies. Let γ be a (local) degree zero syzygy on $J_r(v)$. Rotating Δ if necessary so that no edge is horizontal, we have:

$$J_r(v) = < (x + a_1 \cdot y + b_1 \cdot z)^{r+1}, \ldots, (x + a_n \cdot y + b_n \cdot z)^{r+1} > .$$

So γ expresses a dependence among the rows of the following matrix:

$$\begin{matrix}
1 & \binom{r+1}{1}a_1 & \binom{r+1}{1}b_1 & \binom{r+1}{2}a_1^2 & 2\binom{r+1}{2}a_1b_1 & \binom{r+1}{2}b_1^2 & \ldots & \binom{r+1}{r+1}a_1^{r+1} & (r+1)\binom{r+1}{r+1}a_1^rb_1 & \ldots & \binom{r+1}{r+1}b_1^{r+1} \\
1 & \binom{r+1}{1}a_2 & \binom{r+1}{1}b_2 & \binom{r+1}{2}a_2^2 & 2\binom{r+1}{2}a_2b_2 & \binom{r+1}{2}b_2^2 & \ldots & \binom{r+1}{r+1}a_2^{r+1} & (r+1)\binom{r+1}{r+1}a_2^rb_2 & \ldots & \binom{r+1}{r+1}b_2^{r+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \binom{r+1}{1}a_n & \binom{r+1}{1}b_n & \binom{r+1}{2}a_n^2 & 2\binom{r+1}{2}a_nb_n & \binom{r+1}{2}b_n^2 & \ldots & \binom{r+1}{r+1}a_n^{r+1} & (r+1)\binom{r+1}{r+1}a_n^rb_n & \ldots & \binom{r+1}{r+1}b_n^{r+1}
\end{matrix}$$

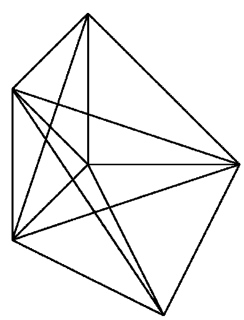
Since γ is a syzygy, it annihilates each column of the matrix. Scaling each column by a nonzero constant will not change the syzygies, so we may drop the binomial coefficients in the matrix above to get a generalized Vandermonde matrix, whose entries are monomials with coefficient 1.

The corresponding matrix for $C^{r-1}(\hat{\Delta})$ consists of a subset of the columns of this matrix (eliminating those columns with monomials in a and b of degree greater than $r-1$). Therefore γ is a syzygy for the smaller matrix also and is contained in $syz(J_{r-1}(v))$. This means $syz(J_r(v))_0 \subset syz(J_{r-1}(v))_0$, so $K_0^r \subset K_0^{r-1}$ and the desired result follows. $\square$

Corollary 2.4 *Let Δ be a topological disk, and fix an r. If dim $C^r_k(\Delta)$ is given by the Alfeld-Schumaker formula, for all k, then for all $q \leq r$, the Alfeld-Schumaker formula also gives the dimension of $C^q_k(\Delta)$.*

3 A counterexample

Rose's conjecture is false in dimension three. Consider a pair of tetrahedra glued along a common triangle, and add an interior vertex in such a way that exactly two of the interior two faces are coplanar. An example is given below; the interior vertex is at $(0,0,0)$, the remaining vertices are placed at $(3,0,0),(0,3,0),(0,0,3),(-2,-2,1),(1,0,-3)$. One can compute that for this configuration, $C^r(\hat{\Delta})$ is free for $r \leq 4$, nonfree for $r = 5$, and free for $r = 6$. A Macaulay2 script to do these computations is available at http://www.math.neu.edu/schenck. The difficulty in seeing this directly is that $C^r(\hat{\Delta})$ is free iff all the lower homology modules of a certain chain complex vanish (see [5], note this implies that topological triviality of Δ is necessary for freeness); in the planar case there is only a single homology module to consider, and (as we saw in the previous section) it has a simple presentation. In higher dimensions, the homology modules become quite complicated; bounds on the dimension of these modules are given in [5], but many open questions remain. Finally, notice that by coning over this example, we also obtain counterexamples in all higher dimensions.



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