Local cohomology of bivariate splines

Hal Schenck [1]

Mathematics Department, Cornell University, Ithaca, NY 14853, U.S.A

Mike Stillman [2]

Mathematics Department, Cornell University, Ithaca, NY 14853, U.S.A

We consider the problem of determining the dimension of the space of bivariate splines $C^r_k(\Delta)$, for all k. This problem is closely related to the question of whether $C^r(\hat{\Delta})$ is a free R-module. The main result is that $C^r(\hat{\Delta})$ is free if and only if $|\Delta|$ has genus zero and $C^r_k(\Delta)$ has the expected dimension for $k = r + 1$ (and hence for all k). We also obtain several interesting corollaries, including the following simple non-freeness criterion: Given a fixed Δ having an edge with both vertices interior, and which does not extend to the boundary, there exists an r_0, which can be determined by inspection, such that $C^r(\hat{\Delta})$ is *not* free for any $r \geq r_0$.

1 Introduction

Let Δ be a connected finite simplicial complex which is supported on $|\Delta| \subset \mathbf{R}^2$. Let $r \geq 0$ be an integer, and let $R = \mathbf{R}[x, y, z]$. Define the spaces of splines

$C^r_k(\Delta) := \{F : |\Delta| \to \mathbf{R} : F|_\sigma$ is a polynomial of degree $\leq k$, for all $\sigma \in \Delta_2$, and F is continuously differentiable of order r$\}$,

and

$C^r(\hat{\Delta}) := \{F : |\hat{\Delta}| \to \mathbf{R} : F|_{\hat{\sigma}} \in R$, for all $\sigma \in \Delta_2$, and F is continuously differentiable of order r$\}$,

[1] Partially supported by the U.S. Army Research Office through the Mathematical Sciences Institute of Cornell University, contract DAAL 03-92-G-0126.
[2] Partially supported by the National Science Foundation through grant DMS 9210805.

where $\hat{\Delta}$ is the join of Δ with the origin in $\mathbf{R}^3$. $C^r(\hat{\Delta})$ is a finitely generated graded R-module.

We address two key problems in spline theory. The first is the determination of the dimensions of $C^r_k(\Delta)$ for all k. The second is to determine whether $C^r(\hat{\Delta})$ is a free R-module. The question of freeness is useful, since if this module is free, then a so-called *reduced* basis of splines exists, in the sense of Billera and Rose ([4]). Freeness is of greatest interest if Δ is supported on a disk, since we show that for all Δ whose support has genus greater than zero, $C^r(\hat{\Delta})$ is not free.

The main theorem presented here (Theorem 4.3) states that $C^r(\hat{\Delta})$ is free if and only if $|\Delta|$ has genus zero and $C^r_k(\Delta)$ has the expected dimension for $k = r+1$ (and hence for all k). We obtain several interesting corollaries from this result and the corresponding techniques. For example, we show that $C^1(\hat{\Delta})$ is free if Δ is generically embedded in the plane. On the other hand, we show that, given a fixed Δ having an edge with both vertices interior, and which does not extend to the boundary, there exists an r_0, which is determined by combinatorial and simple geometric data, such that $C^r(\hat{\Delta})$ is *not* free for any $r \geq r_0$.

Billera introduced the use of homological algebra in spline theory in [3]. In the next section, we recall the homology theory which he introduced. In §3, we define a chain complex $\mathcal{J}$, which depends on r, and present some basic properties of the homology module $H_1(\mathcal{R}/\mathcal{J})$. In §4, we show that this module measures the deviation of the Hilbert series of $C^r(\hat{\Delta})$ from the generic series, and we show that $H_1(\mathcal{R}/\mathcal{J})$ is zero precisely when $C^r(\hat{\Delta})$ is free, and then proceed to prove the results mentioned above. In §5, we present the non-freeness result.

Throughout this paper, a simplicial complex Δ is a connected finite simplicial complex, such that Δ and all its links are pseudomanifolds. Δ^0 denotes the set of interior faces of Δ; Δ_i, Δ_i^0, Δ_i^∂ denote (respectively) the sets of i-dimensional faces, i-dimensional interior faces, and i-dimensional boundary faces. $f_i(\Delta)$, $f_i^0(\Delta)$, and $f_i^\partial(\Delta)$ denote the cardinality of the preceding sets. The genus of Δ is the topological genus, i.e. the rank of the first simplicial homology module. Finally, the complexes $\mathcal{I}$ and $\mathcal{J}$ defined in the next two sections depend on an integer $r \geq 0$, although this is not explicit in the notation.

Computations of the homology modules considered in this paper were generated using a script written in Macaulay II. These methods should generalize to complexes in higher dimensions. We plan to consider these extensions in a further paper.

2 Preliminaries

All the definitions in this section can readily be extended to simplicial complexes embedded in $\mathbf{R}^d$. For simplicity, we restrict to the case where Δ is a simplicial complex in $\mathbf{R}^2$.

Definition 2.1 *Let R be a ring. A complex $\mathcal{F}$ of R-modules on Δ^0 consists of the following data:*

(a) For each $\sigma \in \Delta^0$, an R-module $\mathcal{F}(\sigma)$, and

(b) For each $i \in 0 \ldots d$, an R-module homomorphism $\bigoplus_{\sigma_i \in \Delta_i^0} \mathcal{F}(\sigma_i) \xrightarrow{\partial_i} \bigoplus_{\sigma_{i-1} \in \Delta_{i-1}^0} \mathcal{F}(\sigma_{i-1})$, such that $\partial_{i-1} \circ \partial_i = 0$.

Example 2.2 For any ring R, let $\mathcal{R}$ be the constant complex on Δ^0: $\mathcal{R}(\sigma) = R$, for every $\sigma \in \Delta^0$. Take ∂_i to be the usual (relative to $\partial\Delta$) simplicial boundary map.

Example 2.3 For each $\sigma \in \Delta^0$, let I_σ be the homogeneous ideal of $\hat{\sigma} \subset \mathbf{R}^{d+1}$. I_σ is generated by homogeneous linear polynomials. For example, if $v = (a, b)$ is a vertex of $\Delta \subset \mathbf{R}^2$, then $I_v = (x - az, y - bz) \subset R = \mathbf{R}[x, y, z]$, and if τ is an edge, then I_τ is generated by homogenization of the linear form vanishing on τ.

Fix an integer $r \geq 0$. Define a complex $\mathcal{I}$ of ideals on Δ by $\mathcal{I}(\sigma) := I_\sigma^{r+1}$. Define the quotient complex $\mathcal{R}/\mathcal{I}$ by $\mathcal{R}/\mathcal{I}(\sigma) := R/I_\sigma^{r+1}$.

Given a complex $\mathcal{F}$ of R-modules on Δ:

$$0 \longrightarrow \bigoplus_{\sigma \in \Delta_2} \mathcal{F}(\sigma) \xrightarrow{\partial_2} \bigoplus_{\tau \in \Delta_1^0} \mathcal{F}(\tau) \xrightarrow{\partial_1} \bigoplus_{v \in \Delta_0^0} \mathcal{F}(v) \longrightarrow 0,$$

we define $H_*(\mathcal{F})$ to be the homology of this complex. Given a short exact sequence of complexes:

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

there is a corresponding long exact sequence in homology:

$$0 \to H_2(\mathcal{A}) \to H_2(\mathcal{B}) \to H_2(\mathcal{C}) \to H_1(\mathcal{A}) \to \ldots \to H_0(\mathcal{C}) \to 0.$$

3 The R-modules $H_0(\mathcal{J})$ and $H_1(\mathcal{R}/\mathcal{J})$

Let $R = \mathbf{R}[x,y,z]$, and let Δ be a simplicial complex embedded in the plane.

Fix an integer $r \geq 0$. Define a complex $\mathcal{J}$ by

$$\begin{array}{ll} \mathcal{J}(\sigma) = 0 & \text{for } \sigma \in \Delta_2 \\ \mathcal{J}(\tau) = I_\tau^{r+1} & \text{for } \tau \in \Delta_1^0 \\ \mathcal{J}(v) = \sum_{v \in \tau} I_\tau^{r+1} & \text{for } v \in \Delta_0^0 \end{array}$$

Let $\mathcal{R}$ be the constant complex defined in the previous section, and $\mathcal{R}/\mathcal{J}$ be the quotient of $\mathcal{R}$ by $\mathcal{J}$. By the remarks in §2, the short exact sequence of complexes $0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0$ gives rise to a long exact sequence:

$$0 \to H_2(\mathcal{R}) \to H_2(\mathcal{R}/\mathcal{J}) \to H_1(\mathcal{J}) \to H_1(\mathcal{R}) \to H_1(\mathcal{R}/\mathcal{J}) \to H_0(\mathcal{J}) \to 0$$

It is easy to see that $H_0(\mathcal{R})$ vanishes; so (from the long exact sequence) $H_0(\mathcal{R}/\mathcal{J})$ is also zero. We consider the relationship between the exact sequence of homology modules and the exact sequence of graded modules used by Billera and Rose to define $C^r(\hat{\Delta})$, which we recall below. There is a graded exact sequence:

$$0 \longrightarrow \ker(\phi) \longrightarrow R^{f_2} \bigoplus R^{f_1^0}(-r-1) \xrightarrow{\phi} R^{f_1^0} \longrightarrow \operatorname{coker}(\phi) \longrightarrow 0$$

$$\text{where } \phi = \left(\partial_2 \left| \begin{array}{ccc} l_{\epsilon_1}^{r+1} & & \\ & \ddots & \\ & & l_{\epsilon_{f_1^0}}^{r+1} \end{array} \right. \right).$$

∂_2 is the simplicial (relative to $\partial\Delta$) boundary map from $R^{f_2} \longrightarrow R^{f_1^0}$, and l_{ϵ_i} is the homogeneous linear form defining $I_{\epsilon_i}, \epsilon_i \in \Delta_1^0$. Billera and Rose showed that $C^r(\hat{\Delta})$ is isomorphic to the kernel of ϕ, and also to $H_2(\mathcal{R}/\mathcal{J})$. As it will appear often in the remainder of the paper, we define $M = \operatorname{coker}(\phi)$.

Lemma 3.1 *The homology modules of the complex $\mathcal{R}/\mathcal{J}$ and the module M are related by the following short exact sequence:*

$$0 \longrightarrow H_1(\mathcal{R}/\mathcal{J}) \longrightarrow M \longrightarrow \bigoplus_{v \in \Delta_0^0} \mathcal{R}/\mathcal{J}(v) \longrightarrow 0$$

Proof. Since $H_0(\mathcal{R}/\mathcal{J})$ is zero, the map ∂_1 from $\bigoplus_{\tau \in \Delta_1^0} \mathcal{R}/\mathcal{J}(\tau)$ to $\bigoplus_{v \in \Delta_0^0} \mathcal{R}/\mathcal{J}(v)$ is surjective. We quotient $\bigoplus_{\tau \in \Delta_1^0} \mathcal{R}/\mathcal{J}(\tau)$ by $\operatorname{im}(\partial_2)$ to obtain M. Since $\mathcal{R}/\mathcal{J}$ is a complex, $\operatorname{im}(\partial_2)$ is contained in $\operatorname{kernel}(\partial_1)$, so the resulting map from M to $\bigoplus_{v \in \Delta_0^0} \mathcal{R}/\mathcal{J}(v)$ is still surjective, and by definition has kernel $H_1(\mathcal{R}/\mathcal{J})$. $\square$

It is also worthwhile to note the relationship between the complex $\mathcal{J}$ and the complex $\mathcal{I}$. Alfeld and Schumaker call a vertex v *singular* if $\mathcal{I}(v)/\mathcal{J}(v) \neq 0$. Therefore, for a fixed r, if Δ has no singular interior vertices, then $H_0(\mathcal{J}) = H_0(\mathcal{I})$. The following lemma is the key to the importance of $H_0(\mathcal{J})$.

Lemma 3.2 *The R-module $H_0(\mathcal{J})$ has finite length.*

Proof. For each interior vertex v, and each $f \in \mathcal{J}(v)$, let $f[v]$ be the corresponding element of $H_0(\mathcal{J})$. $H_0(\mathcal{J})$ is generated by $\{f[v] \mid v \in \Delta_0^0$, and $f \in \mathcal{J}(v)\}$. To prove the lemma, it suffices to show that $(x,y,z)^N f[v] = 0$ in $H_0(\mathcal{J})$, for a sufficiently large integer N, for all v, f. For convenience, if v is a boundary vertex, let $f[v] = 0$, for all f.

Choose an ordering $>$ on the interior vertices, and for each vertex v, choose two edges τ' and τ'' with distinct slopes and having equations $l_{\tau'}$ and $l_{\tau''}$ such that if τ' (respectively τ'') has vertices v and w, then either w is a boundary vertex, or $v > w$ in the chosen order. In Lemma 3.3, we prove the existence of such an ordering.

We now argue constructively. Suppose that $(x,y,z)^N g[w] = 0$, for all $w < v$, and all $g \in \mathcal{J}(w)$. Since

$$l_{\tau'}^{r+1} \cdot f[v] = l_{\tau'}^{r+1} \cdot f[w]$$

in $H_0(\mathcal{J})$, and since $l_{\tau'} \in (x,y,z)$, by construction we know that some power of $l_{\tau'}$ annihilates $f[v]$. Similarly, some power of $l_{\tau''}$ annihilates $f[v]$.

Since $f \in \mathcal{J}(v)$, we can write $f = \sum_{\tau \ni v} f_\tau$, where each $f_\tau \in \mathcal{I}(\tau)$. Let $w(\tau)$ be the other vertex of τ. By definition of $H_0(\mathcal{J})$, we have the equation

$$f[v] = \textstyle\sum_{\tau \ni v} f_\tau[w(\tau)],$$

where if $w(\tau)$ is a boundary vertex, then this term is understood to be zero.

Let $p = l_1 \ldots l_k \in R$ be the product of all of the linear forms defining the edges in the link of v. Notice that $p(v) \neq 0$, since none of the forms l_i pass through v, and that $p^{r+1} f[v] = p^{r+1} f[w]$ in $H_0(\mathcal{J})$, for any vertex w in the link of v. Choose w to be the opposite vertex $w(\tau')$ of τ'. By construction, $p^N f[w] = 0$, and therefore $p^N f[v] = 0$.

Thus, some power of each of $l_{\tau'}$, $l_{\tau''}$, and p annihilate $f[v]$. But $(x,y,z)^{N'} \subset (l_{\tau'}, l_{\tau''}, p)$, for some N', and therefore $(x,y,z)^{N'} f[v] = 0$, as desired. □

Lemma 3.3 *If Δ is a triangulated region in $\mathbf{R}^2$, then there exists a total order on Δ_0 such that for every v in Δ_0^0, there exist vertices v', v'' adjacent to v, with $v \succ v'$, v'', and such that $\overline{vv'}$, $\overline{vv''}$ have distinct slopes.*

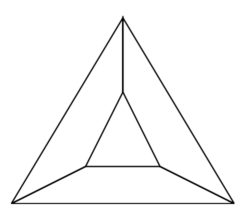
Proof. First, we note that it is enough to prove the lemma for Δ having genus zero, for in the case where Δ has positive genus, we may simply triangulate the holes, find an order which works for the resulting genus zero object, then restrict the order to the original object. Second, we do not require Δ to be connected. Define $d(v, \partial)$ to be the smallest number of edges in a path connecting v to a vertex in the boundary, and $D(\Delta) = \sup_{v \in \Delta_0} \{d(v, \partial)\}$.

We proceed by induction on $D(\Delta)$. If $D(\Delta) = 0$, there is nothing to show. Suppose the lemma holds for Δ such that $D(\Delta) \leq n - 1$, and let Δ be a complex such that $D(\Delta) = n$. Let $S = \{v \in \Delta_0 \mid d(v, \partial) \leq 1\}$. If we can achieve a partial order on S meeting the conditions, then we will be done by the induction hypothesis, because extending the order to the other vertices in Δ_0^0 is equivalent to solving the problem for a Δ' with $D(\Delta') = n - 1$.

Let $v_i \succ v_j$ if $i \succ j$, and assign indices $1 \ldots d$ to the vertices lying on the boundary of Δ ($d = f_0^\partial$). Assign indices $d + 1 \ldots n$ to those interior vertices in S which are joined to the boundary by two or more edges of distinct slope. We need to assign indices to those edges having only one edge (or two edges with the same slope) to the boundary.

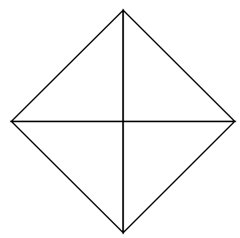
Iterate the following process at each boundary vertex v. Consider the link of v. It is a sequence of vertices and edges, with some vertices indexed (in particular, the 2 vertices at either end of the link are indexed). Each sequence of consecutive unindexed vertices in the link is bounded at the left end by an indexed vertex w. Let k be an integer (which is not already an index) greater than the index of w, and assign it to the leftmost unindexed vertex in the sequence. Repeat. This process terminates with every vertex in the link of v having the desired property. Since every vertex in S is in the link of at least one boundary vertex, we are done. Notice that we need Δ to be simplicial; if Δ is not simplicial, then there may not be two indexed vertices at the leftmost end of the link, which precludes the existence of the desired order. $\square$

Example 3.4 Let Δ correspond to the following complex:



For this (polyhedral) configuration, it is impossible to order the vertices as in Lemma 3.3. For example, if $r = 0$, then from the presentation for $H_0(\mathcal{J})$ given in Lemma 3.8, we find that $H_0(\mathcal{J})$ has three generators and six relations. Of the six relations, three are of degree zero, and three are of degree one. Since the linear forms which vanish on the three interior edges which are not generators intersect in a point, the relations are not independent. In fact, we find that $H_0(\mathcal{J})$ is of the form $R(-1)/l_1, l_2$, where l_1, l_2 are linear forms; so $H_0(\mathcal{J})$ does not have finite length. The simplicial condition is sufficient to insure that this type of situation does not occur.

Example 3.5 This example emphasizes the difference between the complexes $\mathcal{I}$ and $\mathcal{J}$. In particular, notice that $H_0(\mathcal{I})$ need not have finite length. Let Δ correspond to the following complex, and let $r = 1$:



For this configuration, $\mathcal{I}(v) =< x, y >^2$, while the image of ∂_1 is $< x^2, y^2 >$. So $H_0(\mathcal{I}) =< x, y >^2 \;/ < x^2, y^2 >$, and $H_0(\mathcal{I})$ does not have finite length.

Lemma 3.6 *The R-module $H_1(\mathcal{R}/\mathcal{J})$ has finite length.*

Proof. We induct on the rank of $H_1(\mathcal{R})$, i.e. on the genus of Δ. For rank $H_1(\mathcal{R}) = 0$, we have that $H_1(\mathcal{R}/\mathcal{J}) \simeq H_0(\mathcal{J})$, and so are done by Lemma 3.2. Suppose the Lemma is true for all complexes Δ of genus $< g$, and let Δ_g be a complex of genus g. Pick a hole and triangulate it, without modifying the boundary of the hole. This gives us a new complex, Δ_{g-1}, of genus $g-1$. The inclusion of simplicial complexes Δ_g in Δ_{g-1} induces a surjective chain map $\mathcal{R}/\mathcal{J}_{\Delta_{g-1}} \to \mathcal{R}/\mathcal{J}_{\Delta_g}$, and so we have a short exact sequence of complexes:

$$\begin{array}{lcccccc}
\mathcal{K} & : & \bigoplus\limits_{\sigma \in \kappa_2} \mathcal{R} & \longrightarrow & \bigoplus\limits_{\tau \in \kappa_1} \mathcal{R}/\mathcal{J}(\tau) & \longrightarrow & \bigoplus\limits_{v \in \kappa_0} \mathcal{R}/\mathcal{J}(v) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R}/\mathcal{J}_{\Delta_{g-1}} & : & \bigoplus\limits_{\sigma \in (\Delta_{g-1})_2} \mathcal{R} & \longrightarrow & \bigoplus\limits_{\tau \in (\Delta_{g-1})_1^0} \mathcal{R}/\mathcal{J}(\tau) & \longrightarrow & \bigoplus\limits_{v \in (\Delta_{g-1})_0^0} \mathcal{R}/\mathcal{J}(v) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R}/\mathcal{J}_{\Delta_g} & : & \bigoplus\limits_{\sigma \in (\Delta_g)_2} \mathcal{R} & \longrightarrow & \bigoplus\limits_{\tau \in (\Delta_g)_1^0} \mathcal{R}/\mathcal{J}(\tau) & \longrightarrow & \bigoplus\limits_{v \in (\Delta_g)_0^0} \mathcal{R}/\mathcal{J}(v)
\end{array}$$

$\mathcal{K}$ is the kernel complex. κ_2, κ_1, and κ_0 correspond (respectively) to the set of 2-faces of the filled in hole, the union of the set of boundary edges of the filled in hole with the set of interior edges of the filled in hole, and the union of the set of boundary vertices of the hole with the set of interior vertices of the filled in hole. We have a corresponding long exact sequence in homology:

$$\cdots \to H_1(\mathcal{K}) \to H_1(\mathcal{R}/\mathcal{J}_{\Delta_{g-1}}) \to H_1(\mathcal{R}/\mathcal{J}_{\Delta_g}) \to H_0(\mathcal{K}) \to 0$$

(Recall $H_0(\mathcal{R}/\mathcal{J}_\Delta) = 0$ for all Δ connected). By the induction hypothesis, $H_1(\mathcal{R}/\mathcal{J}_{\Delta_{g-1}})$ has finite length, hence so also does its image in $H_1(\mathcal{R}/\mathcal{J}_{\Delta_g})$. Thus, we need only show that $H_0(\mathcal{K})$ has finite length. In $H_0(\mathcal{K})$, all vertices are equivalent, since $\partial(\tau_{ij}) = v_i - v_j$, so since

$$H_0(\mathcal{K}) = \bigoplus_{v\in\kappa_0} \mathcal{R}/\mathcal{J}(v) \Big/ \partial_1(\bigoplus_{\tau\in\kappa_1} \mathcal{R}/\mathcal{J}(\tau)),$$

we have $H_0(\mathcal{K}) = \mathcal{R}/\sum_{v\in\kappa_0} \mathcal{J}(v)$. To see that $H_0(\mathcal{K})$ has finite length, pick any $\sigma \in \kappa_2$, and consider the edges τ_1, τ_2, τ_3 which form its boundary. The $r+1^{st}$ power of each of the three linear forms which vanish on these edges lies in $\mathcal{J}(v_i)$ for some $v_i \in \kappa_0$. But the zero locus of these three forms is (0,0,0), and so the ideal $(l_{\tau_1}^{r+1}, l_{\tau_2}^{r+1}, l_{\tau_3}^{r+1})$ is of codimension three, hence $H_0(\mathcal{K})$ has finite length. □

Remark: It is also possible to prove Lemma 3.6 by a localization argument.

Lemma 3.7 *For all Δ with $H_1(\mathcal{R}) \neq 0$, $H_1(\mathcal{R}/\mathcal{J}) \neq 0$.*

Proof. From the long exact sequence

$$\cdots \longrightarrow H_1(\mathcal{J}) \longrightarrow H_1(\mathcal{R}) \longrightarrow H_1(\mathcal{R}/\mathcal{J}) \longrightarrow \cdots$$

the kernel of the map $H_1(\mathcal{R}) \longrightarrow H_1(\mathcal{R}/\mathcal{J})$ is the image of $H_1(\mathcal{J})$. But $H_1(\mathcal{J})$ is a submodule of $\bigoplus_{\tau\in\Delta_1^0} \mathcal{J}(\tau)$, hence is generated in degree $\geq r+1$. Since $H_1(\mathcal{R})$ is generated in degree zero, if $H_1(\mathcal{R}) \neq 0$, then the map $H_1(\mathcal{J}) \longrightarrow H_1(\mathcal{R})$ cannot be surjective. Therefore, $H_1(\mathcal{R}) \longrightarrow H_1(\mathcal{R}/\mathcal{J})$ is not the zero map, and so $H_1(\mathcal{R}/\mathcal{J}) \neq 0$. □

We call an edge τ *totally interior*, if both vertices of τ are interior.

Lemma 3.8 *Define $K^r \subset \bigoplus_{\tau\in\Delta_1^0} Re_\tau$ to be the submodule generated by*

$$\{e_\tau \mid \tau \textit{ not totally interior}\}$$

and for each $v \in \Delta_0^0$

$$\{\textstyle\sum_{v\in\tau} a_\tau e_\tau \mid \sum_{v\in\tau} a_\tau l_\tau^{r+1} = 0, \textit{for } a_\tau \in R\}.$$

The R-module $H_0(\mathcal{J})$ is given by generators and relations by:

$$0 \longrightarrow K^r \longrightarrow \bigoplus_{\tau\in\Delta_1^0} Re_\tau \longrightarrow H_0(\mathcal{J}) \longrightarrow 0.$$

Proof. Consider the following diagram with exact rows and columns.

$$\begin{array}{ccccc}
\bigoplus_{\tau\in\Delta_1^0}\mathcal{J}(\tau) & \longrightarrow & \bigoplus_{v\in\Delta_0^0}\mathcal{J}(v) & \longrightarrow H_0(\mathcal{J}) \longrightarrow 0 \\
\uparrow & & \uparrow & \\
\bigoplus_{\tau\in\Delta_1^0} R(-r-1) & \xrightarrow{\alpha} & \bigoplus_{\substack{v\in\Delta_0^0,\tau\in\Delta_1^0\\ v\in\tau}} R(-r-1)e_{v\tau} & \\
\uparrow & & \uparrow & \\
0 & & \bigoplus_{v\in\Delta_0^0} K_v & \\
 & & \uparrow & \\
 & & 0, &
\end{array}$$

where K_v is isomorphic to the syzygy module of the elements $\{l_\tau^{r+1} : \tau \ni v\}$, and l_τ is the homogeneous linear equation defining the edge τ.

$H_0(\mathcal{J})$ is therefore generated by elements $e_{v\tau}$ for interior vertices v and interior edges $\tau \ni v$. The relation module on these elements is generated by the image of α and $\bigoplus_{v\in\Delta_0^0} K_v$. The image of α consists of the elements $e_{v\tau}$, where τ is an edge with one interior vertex v, and the other a boundary vertex; and the elements $e_{v\tau} - e_{w\tau}$, for τ an edge with interior vertices v and w. The desired presentation results. $\square$

Corollary 3.9 *The graded R-module $H_0(\mathcal{J})$ is generated by elements of degree $r+1$.* $\square$

4 Conditions for $C^r(\hat{\Delta})$ to be free

Let m be the maximal ideal (x, y, z), and $H_m^0(\_)$ the local cohomology functor ([5]). Recall that for N a graded R-module, $H_m^i(N) = 0$ if $i < depth(N)$ or $i > dim(N)$, and is non-zero for $i = depth(N)$ and $i = dim(N)$. From the short exact sequence

$$0 \longrightarrow H_1(\mathcal{R}/\mathcal{J}) \longrightarrow M \longrightarrow \bigoplus_{v\in\Delta_0^0} \mathcal{R}/\mathcal{J}(v) \longrightarrow 0,$$

we get a long exact sequence in local cohomology

$$0 \longrightarrow H_m^0(H_1(\mathcal{R}/\mathcal{J})) \longrightarrow H_m^0(M) \longrightarrow H_m^0(\bigoplus_{v\in\Delta_0^0} \mathcal{R}/\mathcal{J}(v)) \longrightarrow$$
$$H_m^1(H_1(\mathcal{R}/\mathcal{J})) \longrightarrow \cdots$$

Furthermore, since after a change of variables each $\mathcal{J}(v_i)$ is an ideal in 2 variables, and since at each vertex we have at least two lines, we see that $\mathcal{R}/\mathcal{J}(v_i)$ has projective dimension 2, hence depth 1 by the Auslander-Buchsbaum formula. It is clear that $\dim(\bigoplus_{v\in\Delta_0^0} \mathcal{R}/\mathcal{J}(v)) = 1$. Since local cohomology commutes with direct sums, we see that $H^i_m(\bigoplus_{v\in\Delta_0^0} \mathcal{R}/\mathcal{J}(v)) = 0$ if $i \neq 1$, and since by Lemma 3.6 $H_1(\mathcal{R}/\mathcal{J})$ has finite length, $H^i_m(H_1(\mathcal{R}/\mathcal{J})) = 0$ if $i \neq 0$. Putting this together, we see that the local cohomology of M satisfies:

$H^0_m(M) = H^0_m(H_1(\mathcal{R}/\mathcal{J})) = H_1(\mathcal{R}/\mathcal{J})$

$H^1_m(M) = H^1_m(\bigoplus_{v\in\Delta_0^0} \mathcal{R}/\mathcal{J}(v))$

$H^i_m(M) = 0$ if $i \geq 2$.

Theorem 4.1 *$C^r(\hat{\Delta})$ is free if and only if $H_1(\mathcal{R}/\mathcal{J}) = 0$.*

Proof.

Recall the (graded) exact sequence of section 3:

$$0 \longrightarrow C^r(\hat{\Delta}) \longrightarrow R^{f_2} \oplus R(-r-1)^{f_1^0} \longrightarrow R^{f_1^0} \longrightarrow M \longrightarrow 0,$$

By the Hilbert Syzygy Theorem, the projective dimension of $M \leq 3$, so from the above sequence we see that $C^r(\hat{\Delta})$ is free if and only if M has projective dimension two.

Putting the argument togther,

$$\begin{aligned} H_1(\mathcal{R}/\mathcal{J}) \neq 0 &\iff H^0_m(M) \neq 0 \\ &\iff depth(M) = 0 \\ &\iff pd(M) = 3. \qquad \square \end{aligned}$$

Remark: We have seen in Lemma 3.7 that $H_1(\mathcal{R}/\mathcal{J}) \neq 0$ if $g > 0$, so for the remainder of the paper we restrict our attention to the genus zero case, for that is the only situation where $C^r(\hat{\Delta})$ can be free. We say that a simplicial complex is supported on a disk if it is embedded in $\mathbf{R}^2$ and of genus zero.

The following corollary of Lemma 3.2 follows from Alfeld and Schumaker [1].

Corollary 4.2 *If Δ is a simplicial complex supported on a disk in the plane, and*

$$L(\Delta, r, k) := \dim R_k + f_1^0(\Delta) \dim R_{k-r-1} - \dim \bigoplus_{v\in\Delta_0^0} \mathcal{J}(v)_k,$$

then $\dim C^r_k(\Delta) \geq L(\Delta, r, k)$ for every integer k, and for sufficiently large $k > 0$,

$$\dim C^r_k(\Delta) = L(\Delta, r, k).$$

In fact, equality is achieved for every $k \geq 4r+1$ ([1]). We define $L(\Delta, r, k)$ to be the generic dimension of $C^r_k(\Delta)$, and the generic series to be the formal power series with k^{th} coefficient equal to $L(\Delta, r, k)$.

Proof. First, since $H_1(\mathcal{R}) = 0$, we have the following short exact sequence:

$$0 \longrightarrow H_2(\mathcal{R}) \longrightarrow H_2(\mathcal{R}/\mathcal{J}) \longrightarrow H_1(\mathcal{J}) \longrightarrow 0,$$

so that $C^r(\hat{\Delta}) \simeq R \bigoplus H_1(\mathcal{J})$. From the graded exact sequence

$$0 \longrightarrow H_1(\mathcal{J}) \longrightarrow \bigoplus_{\tau\in\Delta_1^0} \mathcal{J}(\tau) \longrightarrow \bigoplus_{v\in\Delta_0^0} \mathcal{J}(v) \longrightarrow H_0(\mathcal{J}) \longrightarrow 0,$$

it follows that

$$\begin{aligned}\dim C^r_k(\hat{\Delta}) &= \dim R_k + \dim H_1(\mathcal{J})_k \\ &= \dim R_k + \dim \bigoplus_{\tau\in\Delta_1^0} \mathcal{J}(\tau)_k - \dim \bigoplus_{v\in\Delta_0^0} \mathcal{J}(v)_k + \dim H_0(\mathcal{J})_k.\end{aligned}$$

But $\dim \mathcal{J}(\tau)_k = \dim R_{k-r-1}$. Therefore $\dim C^r_k(\hat{\Delta}) = L(\Delta, r, k) + \dim H_0(\mathcal{J})_k$, which implies the first statement. The second statement follows since by Lemma 3.2, $\dim H_0(\mathcal{J})_k = 0$, for sufficiently large k. □

Theorem 4.3 *Let Δ be a simplicial complex supported on a disk in the plane. Let $L(\Delta, r, k)$ be the quantity defined in Corollary 4.2. If $r \geq 0$, then the following are equivalent.*

(a) $C^r(\hat{\Delta})$ is a free R-module;

(b) $\dim C^r_k(\Delta) = L(\Delta, r, k)$ for all integers k;

(c) $\dim C^r_{r+1}(\Delta) = L(\Delta, r, r+1)$.

Proof. By the proof of Corollary 4.2, $\dim C^r(\hat{\Delta})_k = L(\Delta, r, k)$ for all k if and only if $H_0(\mathcal{J}) = 0$, and so the equivalence of (a) and (b) follows by Theorem 4.1. The equivalence of (b) and (c) follows since $H_0(\mathcal{J})$ is generated in degree $r+1$. □

For $r = 1$, the condition for an interior vertex of Δ to be singular is that the number of slopes of the edges incident on the vertex is two.

Corollary 4.4 *Let Δ be a triangulation of a disk in the plane. $C^1(\hat{\Delta})$ is a free R-module if and only if $\dim C^1_2(\Delta) = f^\partial_0(\Delta) + 3 + s$, where s is the number of singular interior vertices.*

Proof. If the interior vertex v is not singular, then after a (homogeneous, linear) change of coordinates, $\mathcal{J}(v) = (x, y)^2$. Therefore

$$\dim \mathcal{J}(v)_k = \binom{k+2}{2} - 3.$$

If v is singular,

$$\dim \mathcal{J}(v)_k = \binom{k+2}{2} - 4.$$

Combining these with the formula for $L(\Delta, 1, k)$, one obtains

$$L(\Delta, 1, k) = \binom{k+2}{2} + f_1^0(\Delta)\binom{k}{2} - f_0^0(\Delta)(\binom{k+2}{2} - 3) + s.$$

In the case that $k = 2$,

$$L(\Delta, 1, 2) = 6 + f_1^0(\Delta) - 3f_0^0(\Delta) + s.$$

Using the Euler formula, and the Dehn-Sommerville equation $3f_2 = 2f_1^0 + f_0^\partial$, one obtains

$$L(\Delta, 1, 2) = f_0^\partial(\Delta) + 3 + s.$$

This proves the forward implication.

Conversely, if $\dim C_2^1(\Delta) = f_0^\partial(\Delta) + 3 + s = L(\Delta, 1, 2)$, then Theorem 4.3 implies that $C^1(\hat{\Delta})$ is free. □

Corollary 4.5 *If Δ is a triangulation of a disk in the plane, and Δ is embedded generically in the plane, then $C^1(\hat{\Delta})$ is a free R-module.*

Proof. If Δ is embedded generically, then Billera and Whiteley ([3], [6]) have shown that $\dim C_2^1(\Delta) = f_0^\partial(\Delta) + 3$, and therefore by the previous result, $C^1(\hat{\Delta})$ is free. □

5 Non-freeness results

In this section we find conditions under which $C^r(\hat{\Delta})$ cannot be a free R-module.

Lemma 5.1 *If $l_1, \ldots, l_k$ are the equations of distinct lines through a point, then for any $d \geq k-1$, $l_1^d, \ldots, l_k^d$ are linearly independent. Furthermore, for $d < k-1$, the space of linear dependencies on $l_1^d, \ldots, l_k^d$ has dimension $k-1-d$.*

Proof. Without loss of generality we may assume that the point is the origin, and that $l_i = x + a_i y$, for distinct numbers a_i. The dimension of the space of linear dependencies on $l_1^d, \ldots, l_k^d$ is k - rank(A), where A is the $k \times (d+1)$ Vandermonde matrix with rows $(1, a_i, a_i^2, \ldots, a_i^d)$. This matrix has full rank, and therefore rank$(A) = \min(k, d+1)$, and the conclusion follows. $\square$

We now need some notation. We define a relation on interior edges by $\epsilon_i \sim \epsilon_j$ if ϵ_i and ϵ_j share a vertex, and have the same slope. Taking the transitive closure of this relation gives us an equivalence relation on interior edges. $\epsilon_i \sim \epsilon_j$ implies (using the presentation for $H_0(\mathcal{J})$ of Lemma 3.8) that $[\epsilon_i] = [\epsilon_j]$ in $H_0(\mathcal{J})$. If $\epsilon_i \sim \epsilon_j$ with ϵ_j not totally interior, then $[\epsilon_i] = [\epsilon_j] = 0$ in $H_0(\mathcal{J})$. We call such an edge a pseudoboundary edge.

Definition 5.2 *For each edge τ of Δ, define*

$$s_\tau := \max\{\textit{number of slopes at } v_i,\ v_i \textit{ a vertex of an edge } \sigma,\ \sigma \sim \tau\}.$$

Define an integer $s(\Delta)$ by

$$s(\Delta) := \min\{s_\tau \mid \tau \textit{ an edge which is not a pseudoboundary edge of } \Delta\}.$$

Theorem 5.3 *If Δ is a triangulation of a disk in the plane, then*

(a) If every edge of Δ is a pseudoboundary edge, then $C^r(\hat{\Delta})$ is free for all r.

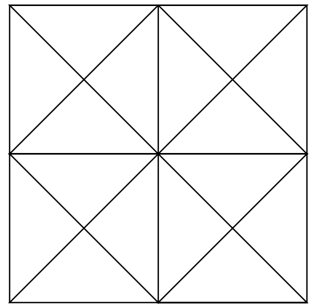
(b) If Δ has at least one edge which is not a pseudoboundary, then for each $r \geq s(\Delta) - 2$, $C^r(\hat{\Delta})$ is not free.

Proof. (a) is immediate, since if every edge is a pseudoboundary edge, then $H_0(\mathcal{J}) = 0$. For (b), let $\tau \in \Delta_1^0$ be an edge for which $s_\tau = s(\Delta)$. If $r+1 \geq s_\tau - 1$, there are no linear (i.e. degree zero) relations at either vertex of the form $0 = \sum_\tau a_\tau l_\tau^{r+1}$, by Lemma 5.1 and the definition of s_τ. Therefore, e_τ only appears in K^r with coefficients of positive degree, so by Lemma 3.8 $H_0(\mathcal{J})_{r+1} \neq 0$. Theorem 4.1 then applies, and $C^r(\hat{\Delta})$ is not free. $\square$

6 Examples

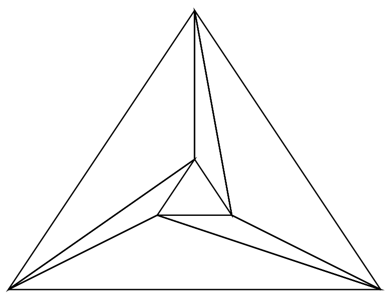
In this section, we give some examples of complexes. The presentation of $H_0(\mathcal{J})$ given in Lemma 3.8 is particularly well-suited to computation.

Example 6.1 Let Δ be the complex corresponding to the following triangulation.



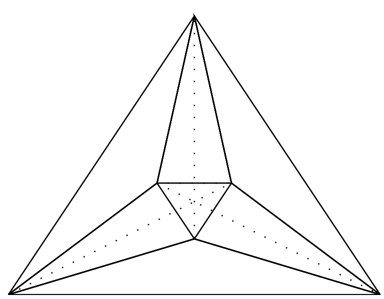
By Theorem 5.3, $C^r(\hat{\Delta})$ is free for all r, since every edge is a pseudoboundary.

Example 6.2 [Morgan-Scott] Let Δ correspond to the following complex.



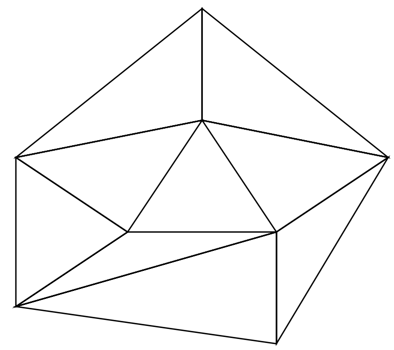
By Theorem 5.3, $C^2(\hat{\Delta})$ is not free. In fact, for $r = 2$, $H_0(\mathcal{J})$ has three generators, and no degree zero relations on these three generators. Therefore the dimension of $C^2_3(\Delta)$ is three larger than the generic dimension 7.

For $r = 1$, $H_0(\mathcal{J})$ is generated by three elements, and there are three degree zero relations on these. In any case, the maximum dimension of $H_0(\mathcal{J})_2$ is one. The dimension is exactly one if and only if the lines joining the boundary vertices to the "opposite" interior vertices meet in a point, as in:



Notice also that $H_0(\mathcal{J})_3 = 0$, and therefore $C^1_k(\Delta)$ has the expected dimension for all $k \geq 3$.

Example 6.3 Theorem 5.3 does not give a sharp bound for when $C^r(\hat{\Delta})$ is free. For example, let Δ correspond to the triangulation:



For $r \geq 3$, Theorem 5.3 applies, and $C^r(\hat{\Delta})$ is not free. For $r = 2$, there are three generators for $H_0(\mathcal{J})$, but only two degree zero conditions. Therefore $H_0(\mathcal{J}) \neq 0$, and $C^2(\hat{\Delta})$ is not free. For $r = 1$, $H_0(\mathcal{J}) = 0$, and therefore $C^1(\hat{\Delta})$ is free.

Acknowledgement

We would like to thank Lou Billera for many helpful conversations, and the mathematics institute at Oberwolfach for inspiring our interest in these problems.

References

[1] P. Alfeld and L. Schumaker, *The dimension of bivariate spline spaces of smoothness r for degree* $d \geq 4r+1$, Constr. Approximation **3**, 189-197 (1987).

[2] P. Alfeld and L. Schumaker, *On the dimension of bivariate spline spaces of smoothness r and degree* $d = 3r+1$, Numer. Math. **57**, 651-661 (1990).

[3] L. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Transactions of the A.M.S., **310**, 325-340 (1988).

[4] L. Billera and L. Rose, *Modules of piecewise polynomials and their freeness*, Mathematische Zeitschrift **209**, 485-497 (1992).

[5] D. Eisenbud, Commutative Algebra with a view towards Algebraic Geometry, Springer Verlag, N.Y. (1995).

[6] W. Whiteley, *A matrix for splines*, Progress in Approximation Theory, Academic Press, Boston MA. (1991).

[7] S. Yuzvinsky, *Modules of Splines on polyhederal complexes*, Mathematische Zeitschrift **210 2**, 245-254 (1992).