A spectral sequence for splines

Hal Schenck *
Mathematics Department
Cornell University
Ithaca, NY 14853
schenck@math.cornell.edu

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Hal Schenck, Mathematics Department, Cornell University, Ithaca N.Y. 14853

Abstract

We define a complex $\mathcal{R}/\mathcal{J}$ of graded modules on a d-dimensional simplicial complex Δ, so that the top homology module of this complex consists of piecewise polynomial functions (splines) of smoothness r on the cone of Δ. In a series of papers ([4], [5], [6]), Billera and Rose used a similar approach to study the dimension of the spaces of splines on Δ, but with a complex substantially different from $\mathcal{R}/\mathcal{J}$. We obtain bounds on the dimension of the homology modules $H_i(\mathcal{R}/\mathcal{J})$, for all $i < d$, and find a spectral sequence which relates these modules to the spline module. We use this to give simple conditions governing the projective dimension of the spline module. We also prove that if the spline module is free, then it is determined entirely by local data; that is, by the arrangements of hyperplanes incident to the various dimensional faces of Δ.

1 Introduction

Let Δ be a connected, finite d-dimensional simplicial complex, supported on $|\Delta| \subset \mathbf{R}^d$, such that Δ and all its links are pseudomanifolds. Let $r \geq 0$ be an integer, and let $R = \mathbf{R}[x_1, \ldots, x_{d+1}]$. Define the spaces of splines

$C^r_k(\Delta) = \{F : |\Delta| \to \mathbf{R} : F|_\sigma$ is a polynomial of degree $\leq k$, for all $\sigma \in \Delta_d$, and F is continuously differentiable of order r$\}$,

and

$C^r(\hat{\Delta}) = \{F : |\hat{\Delta}| \to \mathbf{R} : F|_{\hat{\sigma}} \in R$, for all $\sigma \in \Delta_d$, and F is continuously differentiable of order r$\}$,

where $\hat{\Delta}$ is the join of Δ with the origin in $\mathbf{R}^{d+1}$. $C^r(\hat{\Delta})$ is a finitely generated graded R-module.

The relationship between $C^r(\hat{\Delta})$ and $C^r_k(\Delta)$ is that the elements of $C^r(\hat{\Delta})_k$ are precisely the homogenizations of the elements of $C^r_k(\Delta)$, so to study the dimension of the space $C^r_k(\Delta)$ (a fundamental area of interest in spline theory), it suffices to study the Hilbert series of the module $C^r(\hat{\Delta})$, which is a finitely generated graded module.

In [18] and [19], we considered these problems in the case $d = 2$, and showed that the module $C^r(\hat{\Delta})$ was free if and only if $H_1(\mathcal{R}/\mathcal{J})$ vanished, in which case the Hilbert series for $C^r(\hat{\Delta})$ is determined entirely by the number of edges of distinct slope incident to each interior vertex; i.e. purely by local data. In the case $d = 2$, $C^r(\hat{\Delta})$ can be free only if Δ is a (topological) disk.

The use of homological algebra in spline theory was introduced by Billera in [4]. In

[6], Billera and Rose proved the following local criterion for the freeness of the spline module: $C^r(\Delta)$ is free if and only if $C^r(star(\sigma))$ is free for all faces σ of Δ. In the approach described above, this criterion is of limited use, since $\hat{\Delta}$ is the star of the origin, so we seek other conditions to characterize the freeness of $C^r(\hat{\Delta})$.

One alternate approach appeared in [22], where Yuzvinsky used Cech cohomology to study the projective dimension of the spline module. His approach was to consider the poset defined by the intersections of the affine hulls of the faces of Δ, ordered by reverse inclusion. He topologized this poset by using as a base the set of all faces contained in a given face, and then defined a sheaf on this space and computed the cohomology. His approach handled the complications which arise when considering a polyhedral (rather than simplicial) complex quite nicely. The sheaf that he used for the computations was essentially the same as the complex $\mathcal{R}/\mathcal{I}$ (see §2) used by Billera and Rose; the criterion for the spline module to have a given projective dimension is that certain cohomology modules vanish, for all subcomplexes of the form star(σ), $\sigma \in \Delta_i^0$. This is also a local criterion.

In practice, computing the Cech cohomology can be inefficient. For example, if Δ is a planar simplicial complex which is the star of a vertex, then the Cech complex has length equal to the number of two-dimensional faces. In particular, there may be arbitrarily many homology modules to compute, even for this relatively uncomplicated example. On the other hand, using the complex $\mathcal{R}/\mathcal{J}$, it is simple to prove that for this example, $C^r(\hat{\Delta})$ is always free.

The organization of the paper is as follows: in §2 we review the approach of [4], and introduce our notation. In §3, we define the chain complexes (which depend on r) $\mathcal{J}$ and $\mathcal{R}/\mathcal{J}$. We show that $H_d(\mathcal{R}/\mathcal{J}) \simeq C^r(\hat{\Delta})$, prove that the homology module $H_i(\mathcal{R}/\mathcal{J})$ has dimension $\leq i-1$, for all $i < d$, and relate this result to the Hilbert series of $C^r(\hat{\Delta})$. In §4, we restrict to a topological d-ball, and show that there is a spectral sequence relating $C^r(\hat{\Delta})$ to the modules $H_i(\mathcal{R}/\mathcal{J})$, $i < d$. In §5, we give examples in the trivariate case, and discuss further directions for research.

We will use Δ^0 to denote the set of interior faces of Δ (all d dimensional faces are considered interior); Δ_i, Δ_i^0, Δ_i^∂ denote (respectively) the sets of i-dimensional faces, i-dimensional interior faces, and i-dimensional boundary faces. $f_i(\Delta)$, $f_i^0(\Delta)$, and $f_i^\partial(\Delta)$ denote the cardinality of the preceding sets. Finally, the complexes $\mathcal{I}$ and $\mathcal{J}$ defined in the next two sections depend on an integer $r \geq 0$, although this is not explicit in the notation.

2 Preliminaries

The definitions in this section also appear in [18]; we reproduce them for the sake of completeness.

Definition 2.1 *Let R be a ring. A complex $\mathcal{F}$ of R-modules on Δ^0 consists of the following data:*

(a) For each $\sigma \in \Delta^0$, an R-module $\mathcal{F}(\sigma)$, and

(b) For each $i \in 0 \ldots d$, an R-module homomorphism $\bigoplus_{\sigma_i \in \Delta_i^0} \mathcal{F}(\sigma_i) \xrightarrow{\partial_i} \bigoplus_{\sigma_{i-1} \in \Delta_{i-1}^0} \mathcal{F}(\sigma_{i-1})$, *such that* $\partial_{i-1} \circ \partial_i = 0$. *(in [8], Brion defines such an object as a sheaf on a polyhedral complex.)*

Example 2.2 For any ring R, let $\mathcal{R}$ be the constant complex on Δ^0: $\mathcal{R}(\sigma) = R$, for every $\sigma \in \Delta^0$. Take ∂_i to be the usual (relative to $\partial\Delta$, see [16], p. 47) simplicial boundary map. $H_i(\mathcal{R})$ is the usual relative simplicial homology, with coefficients in R.

Example 2.3 For each $\sigma \in \Delta^0$, let I_σ be the homogeneous ideal of $\hat{\sigma} \subset \mathbf{R}^{d+1}$. I_σ is generated by homogeneous linear polynomials. For example, if $v = (a, b)$ is a vertex of $\Delta \subset \mathbf{R}^2$, then $I_v = (x - az, y - bz) \subset R = \mathbf{R}[x, y, z]$, and if τ is an edge, then I_τ is generated by the homogenization of the linear form vanishing on τ.

Fix an integer $r \geq 0$. Define a complex $\mathcal{I}$ of ideals on Δ by $\mathcal{I}(\sigma) := I_\sigma^{r+1}$. Define the quotient complex $\mathcal{R}/\mathcal{I}$ by $\mathcal{R}/\mathcal{I}(\sigma) := R/I_\sigma^{r+1}$.

Given a complex $\mathcal{F}$ of R-modules on Δ:

$$0 \longrightarrow \bigoplus_{\sigma \in \Delta_d} \mathcal{F}(\sigma) \xrightarrow{\partial_d} \bigoplus_{\tau \in \Delta_{d-1}^0} \mathcal{F}(\tau) \xrightarrow{\partial_{d-1}} \cdots \bigoplus_{v \in \Delta_0^0} \mathcal{F}(v) \longrightarrow 0,$$

we define $H_*(\mathcal{F})$ to be the homology of this complex. Given a short exact sequence of complexes:

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

there is a corresponding long exact sequence in homology:

$$0 \to H_d(\mathcal{A}) \to H_d(\mathcal{B}) \to H_d(\mathcal{C}) \to H_{d-1}(\mathcal{A}) \to \cdots \to H_0(\mathcal{C}) \to 0.$$

In [4], Billera showed that the spline module is isomorphic to $H_d(\mathcal{R}/\mathcal{I})$.

3 The dimension of $H_i(\mathcal{R}/\mathcal{J})$

Fix an integer $r \geq 0$. Define a complex of ideals $\mathcal{J}$ on Δ^0 by

$$
\begin{array}{ll}
\mathcal{J}(\sigma) = 0 & \text{for } \sigma \in \Delta_d \\
\mathcal{J}(\tau) = I_\tau^{r+1} & \text{for } \tau \in \Delta^0_{d-1} \\
\mathcal{J}(\xi) = \sum_{\xi \in \tau} I_\tau^{r+1} & \text{for } \xi \in \Delta^0_{d-2} \\
\vdots & \vdots \\
\mathcal{J}(v) = \sum_{v \in \tau} I_\tau^{r+1} & \text{for } v \in \Delta^0_0
\end{array}
$$

Let $\mathcal{R}$ be the constant complex defined in the previous section, and $\mathcal{R}/\mathcal{J}$ be the quotient of $\mathcal{R}$ by $\mathcal{J}$. By the remarks in §2, the short exact sequence of complexes $0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0$ gives rise to a long exact sequence in homology:

$$\cdots \to H_{i+1}(\mathcal{R}/\mathcal{J}) \to H_i(\mathcal{J}) \to H_i(\mathcal{R}) \to H_i(\mathcal{R}/\mathcal{J}) \to H_{i-1}(\mathcal{J}) \to \cdots$$

Notice that since $\mathcal{J}$ and $\mathcal{I}$ agree on the d and $d-1$ faces, we have $H_d(\mathcal{R}/\mathcal{J}) = H_d(\mathcal{R}/\mathcal{I})$, so that for both complexes the spline module $C^r(\hat{\Delta})$ appears as the top homology module. However, the lower homology modules do indeed differ, and, as we will see, $\mathcal{R}/\mathcal{J}$ has nicer properties than $\mathcal{R}/\mathcal{I}$.

Lemma 3.1 *For all $i < d$, $H_i(\mathcal{R}/\mathcal{J})$ has dimension $\leq i-1$.*

Proof. We make use of the fact that localization commutes with homology. Let 1_α denote the unit of $\mathcal{R}/\mathcal{J}(\alpha)$. We have the complex:

$$\cdots \longrightarrow \bigoplus_{\alpha \in \Delta^0_{i+1}} \mathcal{R}/\mathcal{J}(\alpha) \xrightarrow{\partial_{i+1}} \bigoplus_{\beta \in \Delta^0_i} \mathcal{R}/\mathcal{J}(\beta) \xrightarrow{\partial_i} \bigoplus_{\gamma \in \Delta^0_{i-1}} \mathcal{R}/\mathcal{J}(\gamma) \xrightarrow{\partial_{i-1}} \cdots$$

Let P be a prime ideal such that $P \not\supseteq \mathcal{J}(\gamma)$, for any $\gamma \in \Delta^0_{i-1}$. Then $\bigoplus_{\gamma \in \Delta^0_{i-1}} \mathcal{R}/\mathcal{J}(\gamma)_P = 0$, so $H_i(\mathcal{R}/\mathcal{J})_P = \bigoplus_{\beta \in \Delta^0_i} \mathcal{R}/\mathcal{J}(\beta)_P/(im\partial_{i+1})_P$.

Case 1: $P \not\supseteq \mathcal{J}(\beta)$, for any $\beta \in \Delta^0_i$. Then $\bigoplus_{\beta \in \Delta^0_i} \mathcal{R}/\mathcal{J}(\beta)_P = 0$, so $H_i(\mathcal{R}/\mathcal{J})_P = 0$.

Case 2: $P \supseteq \mathcal{J}(\beta)$, for some (possibly several) $\beta \in \Delta^0_i$. For $\alpha \in \Delta^0_{i+1}$, the map ∂_{i+1} takes 1_α to a signed sum of 1_β, where β is a facet of α. Localization at P sends 1_β to zero if $\mathcal{J}(\beta) \not\subseteq P$. Since two facets of α intersect in a face of dimension $i-1$ (Δ is simplicial), the assumption that P does not contain $\mathcal{J}(\gamma)$ for any $i-1$ face γ implies that in the localization of $\partial_{i+1}(1_\alpha)$, at most one 1_β is nonzero, and some 1_β is nonzero only if $P \supseteq \mathcal{J}(\beta)$. Thus $\partial_{i+1_P}(1_\alpha) = 1_\beta$ if $\beta \subseteq \alpha$ and $\mathcal{J}(\beta) \subseteq P$, so ∂_{i+1_P} is surjective and $H_i(\mathcal{R}/\mathcal{J})_P = 0$.

We have shown that $H_i(\mathcal{R}/\mathcal{J})_P = 0$ if $P \not\supseteq \mathcal{J}(\gamma)$ for any $\gamma \in \Delta^0_{i-1}$. Since P is prime, this means that if $H_i(\mathcal{R}/\mathcal{J})_P \neq 0$, then $P \supseteq I(\gamma)$, for some $\gamma \in \Delta^0_{i-1}$. If we can show that $H_i(\mathcal{R}/\mathcal{J})_{I(\gamma)} = 0$ for all $\gamma \in \Delta^0_{i-1}$, then since $I(\gamma)$ is of codimension $d-i+1$, $H_i(\mathcal{R}/\mathcal{J})$ is supported on primes of codimension at least $d-i+2$, which will conclude the proof.

Suppose $P = I(\gamma)$, some $\gamma \in \Delta^0_{i-1}$. Then the localized complex splits into a direct sum of subcomplexes, of which two types can contribute to $H_i(\mathcal{R}/\mathcal{J})_P$. The first type are those of the following form, with one piece for each $i-1$ face γ_j such that $\hat{\gamma}_j \subseteq V(I(\gamma))$:

$$\cdots \longrightarrow \bigoplus_{\substack{\gamma_j \in \alpha \\ \alpha \in \Delta_{i+1}}} \mathcal{R}/\mathcal{J}(\alpha)_{I(\gamma)} \longrightarrow \bigoplus_{\substack{\gamma_j \in \beta \\ \beta \in \Delta^0_i}} \mathcal{R}/\mathcal{J}(\beta)_{I(\gamma)} \longrightarrow \mathcal{R}/\mathcal{J}(\gamma_j)_{I(\gamma)} \longrightarrow 0$$

The map $\partial_{i_{I(\gamma)}}$ sends each summand surjectively to $\mathcal{R}/\mathcal{J}(\gamma_j)_{I(\gamma)}$, so the kernel of $\partial_{i_{I(\gamma)}}$ is generated by pairs of units with opposite orientations (e.g. $1_{\beta_i} - 1_{\beta_j}$), along with generators of the form $l_{\beta_i}^{r+1} \cdot 1_{\beta_j}$, where $l_{\beta_i}^{r+1} \in \mathcal{J}(\beta_i)$, but $l_{\beta_i}^{r+1} \notin \mathcal{J}(\beta_j)$. An $i+1$ face α in the above subcomplex has a pair of i faces β_i, β_j which intersect in γ_j (again, we make use of the fact that Δ is simplicial), and clearly $\partial_{i+1_{I(\gamma)}}(1_\alpha) = 1_{\beta_i} - 1_{\beta_j}$, which generate all elements of the kernel of the first type mentioned above. For generators of the second type, notice that modulo the image of $\partial_{i+1_{I(\gamma)}}$, $l_{\beta_i}^{r+1} \cdot 1_{\beta_j} = l_{\beta_i}^{r+1} \cdot 1_{\beta_i}$, so is zero in homology. Thus, these subcomplexes do not contribute to $H_i(\mathcal{R}/\mathcal{J})_P$.

The second type of subcomplex which may contribute to $H_i(\mathcal{R}/\mathcal{J})_P$ are those with $V(I(\gamma)) \subseteq V(I(\beta_k))$, $\beta_k \in \Delta_i^0$, but where β_k does not contain an $i-1$ face γ_k such that $\hat{\gamma}_k \subseteq V(I(\gamma))$. These complexes take the form:

$$\cdots \longrightarrow \bigoplus_{\substack{\beta_k \in \alpha \\ \alpha \in \Delta_{i+1}}} \mathcal{R}/\mathcal{J}(\alpha)_{I(\gamma)} \longrightarrow \mathcal{R}/\mathcal{J}(\beta_k)_{I(\gamma)} \longrightarrow 0$$

It is easy to check that the localized ∂_{i+1} map is surjective, and hence for these subcomplexes we also have $H_i(\mathcal{R}/\mathcal{J})_P = 0$. □

Lemma 3.1 illustrates the difference between the complexes $\mathcal{R}/\mathcal{I}$ and $\mathcal{R}/\mathcal{J}$. In particular, Lemma 3.1 is false for $\mathcal{R}/\mathcal{I}$. The preceding proof fails because on the complex $\mathcal{R}/\mathcal{I}$ the kernel of $\partial_{i_{I(\gamma_j)}}$ (∂_i on both $\mathcal{R}/\mathcal{J}$ and $\mathcal{R}/\mathcal{I}$ is induced by the simplicial (relative) boundary map on $\mathcal{R}$) also contains generators of the form $f \cdot 1_{\beta_i}$, where $f \in \mathcal{I}(\gamma_j)$. Such an element need not be in the image of $\partial_{i+1_{I(\gamma_j)}}$, and so in general it is not the case that $H_i(\mathcal{R}/\mathcal{I})_{I(\gamma)} = 0$, $\gamma \in \Delta_{i-1}^0$. A specific example of this, for the case $d = 2$, is given in §3 of [18].

Examples (see §5) show that it is possible for $H_i(\mathcal{R}/\mathcal{J})$ to be non-zero, but to have support on primes of codimension $> d-i+2$. It would be interesting to know if these primes are somehow reflecting properties of lower dimensional subcomplexes of Δ; or of a projection or hyperplane section.

Corollary 3.2 *The Hilbert series of the spline module is dominated by the behavior of the modules* $\bigoplus_{\beta\in\Delta_i^0} \mathcal{R}/\mathcal{J}(\beta)$, *for* $i = d, d-1, d-2$.

Proof. Let $P(N,t)$ be the numerator of the Hilbert series for the graded module N. Recall the Euler characteristic equation:

$$\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J}).$$

Thus, we can write

$$P(H_d(\mathcal{R}/\mathcal{J}),t) = \sum_{i=0}^{d}(-1)^i P(\bigoplus_{\beta\in\Delta_{d-i}^0} \mathcal{R}/\mathcal{J}(\beta),t) + \sum_{i=0}^{d-1}(-1)^i P(H_{d-i-1}(\mathcal{R}/\mathcal{J}),t).$$

Since the homology modules in the second summation have dimension at most $d-2$, as do the modules $\bigoplus_{\beta\in\Delta_i^0} \mathcal{R}/\mathcal{J}(\beta)$, $i \le d-3$, the result follows. The point is that the objects $\bigoplus_{\beta\in\Delta_i^0} \mathcal{R}/\mathcal{J}(\beta)$, $i = d, d-1$ are trivial to understand, and if $i = d-2$, then we have a set of hyperplanes intersecting in a codimension two linear subspace, so we may reduce to the bivariate case, which was considered in [19]. In particular, if β is a face of dimension $d-2$, such that there are k distinct hyperplanes incident to β, then the numerator of the Hilbert series of $\mathcal{R}/\mathcal{J}(\beta)$ is given by:

$$1 - kt^{r+1} + (k-1)\lambda t^{r+1+\lambda} + (r+1+(k-1)\lambda)t^{r+2+\lambda},$$

where $\lambda = \lfloor \frac{r+1}{k-1} \rfloor$. $\square$

4 The projective dimension of $C^r(\hat{\Delta})$

In [18], we showed that in the two-dimensional case, $C^r(\hat{\Delta})$ can be free only if Δ is a (topological) disk. For the remainder of the paper we restrict to the case of Δ a topological d-ball, i.e. $H_i(\mathcal{R}) = 0$, $i \neq d$. Thus, $C^r(\hat{\Delta}) \simeq R \oplus H_{d-1}(\mathcal{J})$, and $H_i(\mathcal{R}/\mathcal{J}) \simeq H_{i-1}(\mathcal{J})$ for all $i \leq d-1$ (hence by Lemma 3.1, $H_i(\mathcal{J})$ has dimension less than or equal to i for $i < d-1$). In particular, the study of the spline module reduces to the study of $H_{d-1}(\mathcal{J})$. We begin by collecting some lemmas which we will need in this section.

Lemma 4.1 *(Eisenbud, Huneke, and Vasconcelos [12]) Let M be a graded module over the polynomial ring R. Then the annihilator of $Ext^e(M,R)$ has codimension at least e, and a prime ideal P of codimension e is associated to M iff P contains the annihilator of $Ext^e(M,R)$.*

Lemma 4.2 *(Ischebeck, [14]) If A,m is a Noetherian local ring, M,N finite A-modules with $depth(N) = k$ and $dim(M) = r$, then $Ext^i(M,N) = 0$ for all $i < k - r$.*

Lemma 4.3 *$Ext^j(\bigoplus_{\beta \in \Delta_i^0} \mathcal{J}(\beta), R)$ is zero if $j \neq 0$ or $j \neq d - i - 1$.*

Proof. The short exact sequence

$$0 \longrightarrow \bigoplus_{\beta \in \Delta_i^0} \mathcal{J}(\beta) \to \bigoplus_{\beta \in \Delta_i^0} \mathcal{R}(\beta) \longrightarrow \bigoplus_{\beta \in \Delta_i^0} \mathcal{R}/\mathcal{J}(\beta) \longrightarrow 0$$

yields a long exact sequence of Ext modules. Since $Hom(\bigoplus_{\beta \in \Delta_i^0} \mathcal{R}/\mathcal{J}(\beta), R) = 0$ and $Ext^j(\bigoplus_{\beta \in \Delta_i^0} \mathcal{R}(\beta), R) \neq 0 \leftrightarrow j = 0$, we obtain $Hom(\bigoplus_{\beta \in \Delta_i^0} \mathcal{J}(\beta), R) \neq 0$, and

$Ext^j(\bigoplus_{\beta\in\Delta_i^0} \mathcal{J}(\beta), R) = Ext^{j+1}(\bigoplus_{\beta\in\Delta_i^0} \mathcal{R}/\mathcal{J}(\beta), R)$, $j \geq 1$. We need to show that if $j \geq 2$, then $j \neq d-i$ implies that $Ext^j(\bigoplus_{\beta\in\Delta_i^0} \mathcal{R}/\mathcal{J}(\beta), R)$ is zero.

$\mathcal{J}(\beta)$ is an $I(\beta)$-primary ideal, so $\mathcal{J}(\beta)$ is of codimension $d-i$, hence Lemma 4.2 implies $Ext^j(\bigoplus_{\beta\in\Delta_i^0} \mathcal{R}/\mathcal{J}(\beta), R) = 0$ for all $j < d-i$.

For $\beta \in \Delta_i^0$, since $\mathcal{J}(\beta)$ cuts out the linear variety $\hat{\beta}$, after a change of variables, we may assume that $\mathcal{J}(\beta)$ is an ideal involving only $d-i$ variables, so that by the Hilbert syzygy theorem, the projective dimension of $\bigoplus_{\beta\in\Delta_i^0} \mathcal{R}/\mathcal{J}(\beta)$ is at most $d-i$, and hence $Ext^j(\bigoplus_{\beta\in\Delta_i^0} \mathcal{R}/\mathcal{J}(\beta), R) = 0$, $j > d-i$. $\Box$

Lemma 4.4 *If $p < d-1$ and $Ext^q(H_p(\mathcal{J}), R)$ is nonzero, then $q \geq d+1-p$.*

Proof. If $N = R$, then Lemma 4.2 specializes to $Ext^i(M,R) = 0$ for $i < d+1-dim(M)$. By Lemma 3.1, $H_p(\mathcal{J})$ has dimension at most p, so the result follows. Again notice the difference between the complexes $\mathcal{I}$ and $\mathcal{J}$; it is not true that the only possible nonvanishing $Ext^q(H_p(\mathcal{I}), R)$ are those with $q \geq d+1-p$. $\Box$

Lemma 4.5 *If $0 \to K \to M \to N \to 0$ is a short exact sequence, and if M has associated primes of codimension e, while N has only associated primes of codimension greater than e, then K has associated primes of codimension e.*

Proof. Since $Ass(K) \subseteq Ass(M) \subseteq Ass(N) \cup Ass(K)$ (see [11]), the result follows. $\Box$

Example 4.6 Suppose Δ is three dimensional (i.e. $d = 3$), and apply the functor Ext to the short exact sequences of the following form, for $i = 0, 1, 2$:

$$0 \longrightarrow H_i(\mathcal{J}) \longrightarrow \bigoplus_{\beta \in \Delta_i^0} \mathcal{J}(\beta)/im(\partial_{i+1}) \longrightarrow im(\partial_i) \longrightarrow 0$$

$$0 \longrightarrow im(\partial_i) \longrightarrow \bigoplus_{\gamma \in \Delta_{i-1}^0} \mathcal{J}(\gamma) \longrightarrow \bigoplus_{\gamma \in \Delta_{i-1}^0} \mathcal{J}(\gamma)/im(\partial_i) \longrightarrow 0,$$

If $C^r(\hat{\Delta})$ is free, then Lemma 3.1 and a bit of work with the long exact sequences of Ext modules obtained from the above short exact sequences shows that $H_1(\mathcal{J}) = H_0(\mathcal{J}) = 0$. The converse is also easy to prove, so that for $d = 3$, $C^r(\hat{\Delta})$ free if and only if $H_1(\mathcal{J}) = H_0(\mathcal{J}) = 0$. Using the same approach as above, in the case $d = 4$ it is possible to prove that if $C^r(\hat{\Delta})$ is free, then $Ext^j(H_i(\mathcal{J}), R) = 0$ for $i \leq 2$, all j, with the exception of $Ext^3(H_2(\mathcal{J}), R)$ and $Ext^5(H_1(\mathcal{J}), R)$, which must be isomorphic.

This isomorphism suggests that there may be a spectral sequence ([7], [9], [11]) present; computing the hyperhomology of the complex $\mathcal{J}$, we find that this is indeed the case. In order to keep the presentation as self-contained as possible, we will go through the steps in some detail. Take a Cartan-Eilenberg resolution for the complex $\mathcal{J}$. Recall that this means that we first take free resolutions $F_{i+1,\bullet}$ for $im(\partial_{i+1})$ and $G_{i,\bullet}$ for $H_i(\mathcal{J})$. Then the short exact sequence

$$0 \longleftarrow H_i(\mathcal{J}) \longleftarrow ker(\partial_i) \longleftarrow im(\partial_{i+1}) \longleftarrow 0$$

allows us to take the direct sum $G_{i,\bullet} \oplus F_{i+1,\bullet}$ as a resolution for $ker(\partial_i)$.

Similarly, from the short exact sequence

$$0 \longleftarrow im(\partial_i) \longleftarrow \bigoplus_{\beta \in \Delta_i^0} \mathcal{J}(\beta) \longleftarrow ker(\partial_i) \longleftarrow 0,$$

we obtain a resolution for $\bigoplus_{\beta\in\Delta_i^0} \mathcal{J}(\beta)$ from the resolutions of $ker(\partial_i)$ and $im(\partial_i)$. To be specific, the j^{th} module in the free resolution of $\bigoplus_{\beta\in\Delta_i^0} \mathcal{J}(\beta)$ is $F_{i+1,j} \oplus G_{i,j} \oplus F_{i,j}$. This gives us a double complex, with the obvious vertical differential, and with horizontal differential

$$F_{i,j} \oplus G_{i-1,j} \oplus F_{i-1,j} \xleftarrow{d_{i,j}} F_{i+1,j} \oplus G_{i,j} \oplus F_{i,j}.$$

The key fact is that $d_{i,j}$ is the identity on $F_{i,j}$ and zero elsewhere. Applying the functor $Hom(-,R)$ to this double complex, we obtain a new double complex. If we first compute the total homology of the complex using the vertical filtration (this means compute the homology with respect to the vertical differentials), then we obtain

$$E^1_{p,q} = Ext^q(\bigoplus_{\beta\in\Delta_p^0} \mathcal{J}(\beta), R).$$

This is nonzero only if $q = 0$ or $q = d - p - 1$, by Lemma 4.3. Thus, when we compute $E^j_{p,q}$ for $j > 1$, the only possible nonvanishing terms are those with $p + q \leq d - 1$. In particular, $E^\infty_{p,q} = 0$ for $p + q \geq d$.

We pause for a moment to recall the "fundamental theorem" of spectral sequences. Given a first quadrant double complex, we can make it into a single complex in the obvious fashion. Again, to be completely precise, if the double complex is given by $G_{p,q}$, then we put $G'_l = \bigoplus_{p+q=l} G_{p,q}$, and set the differential to be $d'_l := d_{vert} + (-1)^l d_{hor}$. Then the homology of this complex is a filtered module, and $\bigoplus_{p+q=n} E^\infty_{p,q}$ is the direct sum of the quotients in a filtration of the n^{th} homology of the total complex. The point is that the two different filtrations (vertical and horizontal) both give filtrations of the same object, so that if one filtration has $\bigoplus_{p+q=n} E^\infty_{p,q} = 0$, then this is also true for the other filtration. We will use this to exploit the fact that for the vertical filtration, the only possible nonvanishing $E^{\infty}_{p,q}$ are those with $p+q \leq d-1$. By comparing these terms to the E^{∞} terms of the horizontal filtration (a standard use of spectral sequences), we will obtain our results.

Example 4.7 We compute an example for the case $d = 4$. The original complex is:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
F_{1,3} \oplus G_{0,3} & \leftarrow & F_{2,3} \oplus G_{1,3} \oplus F_{1,3} & \leftarrow & F_{3,3} \oplus G_{2,3} \oplus F_{2,3} & \leftarrow & G_{3,3} \oplus F_{3,3} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_{1,2} \oplus G_{0,2} & \leftarrow & F_{2,2} \oplus G_{1,2} \oplus F_{1,2} & \leftarrow & F_{3,2} \oplus G_{2,2} \oplus F_{2,2} & \leftarrow & G_{3,2} \oplus F_{3,2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_{1,1} \oplus G_{0,1} & \leftarrow & F_{2,1} \oplus G_{1,1} \oplus F_{1,1} & \leftarrow & F_{3,1} \oplus G_{2,1} \oplus F_{2,1} & \leftarrow & G_{3,1} \oplus F_{3,1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_{1,0} \oplus G_{0,0} & \leftarrow & F_{2,0} \oplus G_{1,0} \oplus F_{1,0} & \leftarrow & F_{3,0} \oplus G_{2,0} \oplus F_{2,0} & \leftarrow & G_{3,0} \oplus F_{3,0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v \in \Delta_0^0} \mathcal{J}(v) & \leftarrow & \bigoplus_{\epsilon \in \Delta_1^0} \mathcal{J}(\epsilon) & \leftarrow & \bigoplus_{\tau \in \Delta_2^0} \mathcal{J}(\tau) & \leftarrow & \bigoplus_{\sigma \in \Delta_3^0} \mathcal{J}(\sigma)
\end{array}$$

(The complex $\mathcal{J}$ is included in the diagram for clarity, it is not involved in the homology computations). Dualizing and computing the vertical homology, we obtain:

$$\begin{array}{cccc}
Ext^3(\bigoplus_{v \in \Delta_0^0} \mathcal{J}(v), R) & 0 & 0 & 0 \\
0 & Ext^2(\bigoplus_{\epsilon \in \Delta_1^0} \mathcal{J}(\epsilon), R) & 0 & 0 \\
0 & 0 & Ext^1(\bigoplus_{\tau \in \Delta_2^0} \mathcal{J}(\tau), R) & 0 \\
Hom(\bigoplus_{v \in \Delta_0^0} \mathcal{J}(v), R) & Hom(\bigoplus_{\epsilon \in \Delta_1^0} \mathcal{J}(\epsilon), R) & Hom(\bigoplus_{\tau \in \Delta_2^0} \mathcal{J}(\tau), R) & Hom(\bigoplus_{\sigma \in \Delta_3^0} \mathcal{J}(\sigma), R)
\end{array}$$

Now consider the horizontal filtration (so we first compute the homology using the horizontal differentials). Since dualizing the complex corresponds to transposing the differentials, and since the transpose of the identity is the identity, we see that $d^{*}_{i,j}$ is the identity on $F^*_{i,j}$ and zero elsewhere, so the kernel of $d^*_{i,j}$ is $F^*_{i-1,j} \oplus G^*_{i-1,j}$, and the image of $d^*_{i-1,j}$ is $F^*_{i-1,j}$. Thus, for the horizontal filtration,

$$E^1_{p,q} = G^*_{p,q}.$$

That is, the horizontal homology at position p, q is just the dual module of the q^{th} module in the free resolution of $H_p(\mathcal{J})$. Computing the E^2 terms with the horizontal filtration (the vertical differential induces a differential on the E^1 terms, and the E^2 terms are the homology with respect to this differential), we find that for the horizontal filtration,

$$E^2_{p,q} = Ext^q(H_p(\mathcal{J}), R).$$

By Lemma 4.4, for $p < d-1$ the only possible nonvanishing $Ext^q(H_p(\mathcal{J}), R)$ are those with $q \geq d+1-p$, so we have the following diagram for the terms $E^2_{p,q}$ of the horizontal filtration:

$Ext^5(H_0(\mathcal{J}), R)$	$Ext^5(H_1(\mathcal{J}), R)$	$Ext^5(H_2(\mathcal{J}), R)$	$Ext^5(H_3(\mathcal{J}), R)$
0	$Ext^4(H_1(\mathcal{J}), R)$	$Ext^4(H_2(\mathcal{J}), R)$	$Ext^4(H_3(\mathcal{J}), R)$
0	0	$Ext^3(H_2(\mathcal{J}), R)$	$Ext^3(H_3(\mathcal{J}), R)$
0	0	0	$Ext^2(H_3(\mathcal{J}), R)$
0	0	0	$Ext^1(H_3(\mathcal{J}), R)$
0	0	0	$Hom(H_3(\mathcal{J}), R)$

For the horizontal filtration, the d^2 differential takes $E^2_{p,q}$ to $E^2_{p-1,q+2}$. If we suppose that $C^r(\hat{\Delta})$ is free, then $Ext^i(H_3(\mathcal{J}), R) = 0$, $i > 0$. The kernel of the map from

$Ext^3(H_2(\mathcal{J}), R)$ to $Ext^5(H_1(\mathcal{J}), R)$ is $E^3_{2,3}$, and the cokernel is $E^3_{1,5}$. For this example, it is easy to see that the E^3 terms must be stable (recall, for the horizontal filtration, d^i takes $E^i_{p,q}$ to $E^i_{p-i+1,q+i}$). Comparing them to the terms of the vertical filtration, we find that they both vanish, so the map from $Ext^3(H_2(\mathcal{J}), R)$ to $Ext^5(H_1(\mathcal{J}), R)$ is an isomorphism, while all the other $Ext^j(H_i(\mathcal{J}), R)$ with $j > 0$ must vanish. This agrees with the result we obtained from the short exact sequences, but with much less effort.

If $H_2(\mathcal{J})$ has codimension greater than three, then $Ext^3(H_2(\mathcal{J}), R)$ is zero by Lemma 4.2, which forces $Ext^5(H_1(\mathcal{J}), R)$ to be zero. If $H_2(\mathcal{J})$ has codimension three (the smallest possible codimension, by Lemma 3.1), then by Lemma 4.1, $Ext^3(H_2(\mathcal{J}), R)$ has associated primes of codimension three, while $Ext^5(H_1(\mathcal{J}), R)$ is supported on an ideal of codimension at least five, which contradicts the fact that $Ext^3(H_2(\mathcal{J}), R) \simeq Ext^5(H_1(\mathcal{J}), R)$. We have shown that if $C^r(\hat{\Delta})$ is free, then all the modules $Ext^j(H_i(\mathcal{J}), R)$ vanish, $i < d-1$, which implies all the modules $H_i(\mathcal{J})$ vanish for $i < d-1$. So for the example $d = 4$, we again obtain that $C^r(\hat{\Delta})$ free implies $H_i(\mathcal{J}) = 0, i < d-1 = 3$. This is suggestive of a general result, which we will now prove.

Lemma 4.8 *Let Δ be a complex such that $H_i(\mathcal{R}) = 0$ for all $i < d$. If $C^r(\hat{\Delta})$ is free, then $H_i(\mathcal{J})$ vanishes for all $i < d-1$.*

Proof. If $C^r(\hat{\Delta})$ is free, then $Ext^i(H_{d-1}(\mathcal{J}), R) = 0$ for all $i > 0$, and by Lemma 4.4 $Ext^q(H_p(\mathcal{J}), R) \neq 0 \Rightarrow q \geq d + 1 - p$, so the E^2 terms of the horizontal filtration are:

$$
\begin{array}{ccccc}
Ext^{d+1}(H_0(\mathcal{J}),R) & Ext^{d+1}(H_1(\mathcal{J}),R) & \cdots & Ext^{d+1}(H_{d-2}(\mathcal{J}),R) & 0 \\
0 & Ext^{d}(H_1(\mathcal{J}),R) & \cdots & \vdots & \vdots \\
\vdots & 0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & 0 & Ext^{3}(H_{d-2}(\mathcal{J}),R) & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Hom(H_{d-1}(\mathcal{J}),R)
\end{array}
$$

Since $Ext^1(H_{d-1}(\mathcal{J}),R)=0$, no differential d^i ever reaches $Ext^{d+1}(H_0(\mathcal{J}),R)$, so $Ext^{d+1}(H_0(\mathcal{J}),R)$ is stable. Comparing it to the $E^{\infty}_{p,q}$ terms for the vertical filtration, which are zero if $p+q\geq d$, we see that it is zero.

We use induction on i to prove that $Ext^{d+1-i}(H_i(\mathcal{J}),R)=0$, i.e. that the lowest diagonal of E^2 terms of the horizontal filtration vanishes. Once we have shown this, then iterating the argument proves that all the $E^2_{p,q}$ terms with $p+q\geq d+1$ vanish, and will conclude the proof of the lemma.

So, suppose $Ext^{d+1-i}(H_i(\mathcal{J}),R)=0$ for all $i<k$, and let $i=k$. If $H_k(\mathcal{J})$ has codimension greater than $d+1-k$, then by Lemma 4.2 $Ext^{d+1-k}(H_k(\mathcal{J}),R)=0$, so we assume that $H_k(\mathcal{J})$ has codimension $d+1-k$. The map $d^2_{k,d+1-k}$ takes $Ext^{d+1-k}(H_k(\mathcal{J}),R)$ to $Ext^{d+3-k}(H_{k-1}(\mathcal{J}),R)$. The image of $d^2_{k,d+1-k}$ has codimension at least $d+3-k$ (it is contained in $Ext^{d+3-k}(H_{k-1}(\mathcal{J}),R)$, which has codimension at least $d+3-k$, by Lemma 4.1), so since $Ext^{d+1-k}(H_k(\mathcal{J}),R)$ has codimension $d+1-k$, Lemma 4.5 implies that the kernel of $d^2_{k,d+1-k}$ has codimension $d+1-k$, i.e. $E^3_{k,d+1-k}$

has codimension $d+1-k$.

Now, $d^3_{k,d+1-k}$ takes $E^3_{k,d+1-k}$ to $E^3_{k-2,d+4-k}$. The latter module is the kernel of the map from $Ext^{d+4-k}(H_{k-2}(\mathcal{J}), R)$ to $Ext^{d+6-k}(H_{k-3}(\mathcal{J}), R)$, so $E^3_{k-2,d+4-k}$ must have codimension at least $d+4-k$, and again Lemma 4.5 implies that the kernel of $d^3_{k,d+1-k}$ has codimension $d+1-k$, and thus so does $E^4_{k,d+1-k}$.

In general, the image of $d^i_{k,d+1-k}$ is contained in iterated kernels of maps from modules of codimension greater than $d+1-k$ (this is where we use the induction hypothesis, because if $Ext^{d+1-i}(H_i(\mathcal{J}), R) \neq 0$ for some $i < k$, then we don't simply have iterated kernels, i.e. there will be images to quotient out, so we cannot make this statement) and so the $E^\infty_{k,d+1-k}$ term for the horizontal filtration has codimension $d+1-k$. For the vertical filtration, the $E^\infty_{p,q}$ terms vanish for all p,q such that $p+q \geq d$, which implies that $E^\infty_{k,d+1-k}$ vanishes for the horizontal filtration. This means that $Ext^{d+1-k}(H_k(\mathcal{J}), R)$ is supported in codimension greater than $d+1-k$, so by Lemma 4.2 $Ext^{d+1-k}(H_k(\mathcal{J}), R) = 0$. By induction $Ext^{d+1-i}(H_i(\mathcal{J}), R) = 0$, and we are done. □

Lemma 4.9 *Let Δ be a complex such that $H_i(\mathcal{R}) = 0$ for all $i < d$. If $H_i(\mathcal{J}) = 0$ for all $i < d-1$, then $C^r(\hat{\Delta})$ is free.*

Proof. If the $H_i(\mathcal{J})$ vanish for all $i < d-1$, then the $E^2_{p,q}$ terms for the horizontal filtration are all stable, and $E^2_{p,q} = 0$ if $p \neq d-1$. Comparing these terms to the terms from the vertical filtration, we see that in fact $E^2_{p,q} = 0$ except at position $d-1, 0$, so that $H_{d-1}(\mathcal{J})$ is free. □

Theorem 4.10 *If Δ is a complex such that $H_i(\mathcal{R}) = 0$ for all $i < d$, then $C^r(\hat{\Delta})$ is free if and only if $H_i(\mathcal{J}) = 0$ for all $i < d-1$.*

Proof. Immediate from Lemma 4.8 and Lemma 4.9. □

Notice that for the vertical filtration the terms $E^2_{d-1,d}$ and $E^2_{d-1,d+1}$ are stable, hence zero, and so the projective dimension of $C^r(\hat{\Delta})$ is at most $d-1$. More substantially, using the techniques of the preceding proofs, we may generalize Lemma 4.9 to:

Lemma 4.11 *Let Δ be a complex such that $H_i(\mathcal{R}) = 0$ for all $i < d$. If $Ext^j(H_i(\mathcal{J}), R) = 0$ for all i, j such that $i + j \geq d + 1 + k$, $i < d - 1$, then the projective dimension of $C^r(\hat{\Delta})$ is less than or equal to k.*

Corollary 4.12 *If $C^r(\hat{\Delta})$ is free, then the Hilbert series for $C^r(\hat{\Delta})$ is determined entirely by local data; i.e. by the Hilbert series of the various $\mathcal{R}/\mathcal{J}(\sigma), \sigma \in \Delta_i^0$.*

Proof. By Lemma 4.8, if $C^r(\hat{\Delta})$ is free then $H_i(\mathcal{J})$ vanishes for all $i < d-1$, and so $H_i(\mathcal{R}/\mathcal{J})$ vanishes for all $i < d$. Then the formula of Corollary 3.2 specializes to:

$$P(H_d(\mathcal{R}/\mathcal{J}), t) = \sum_{i=0}^{d} (-1)^i P\Big(\bigoplus_{\beta \in \Delta^0_{d-i}} \mathcal{R}/\mathcal{J}(\beta), t\Big).$$

The converse implication may not hold, because the modules $H_i(\mathcal{R}/\mathcal{J})$ can have embedded components, which could lead to cancellation. □

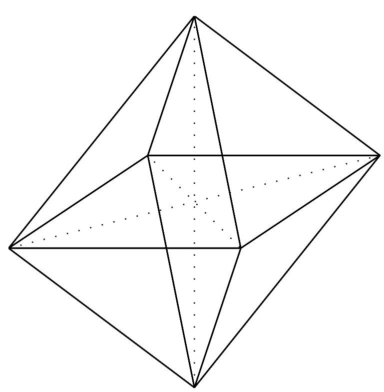
Corollary 4.12 suggests studying the modules $\bigoplus_{\beta \in \Delta_i^0} \mathcal{R}/\mathcal{J}(\beta)$, β a face of codimension greater than two. In [19] we proved that if β is a face of codimension two, then the resolution of the module $\bigoplus_{\beta \in \Delta_i^0} \mathcal{R}/\mathcal{J}(\beta)$ is determined entirely by r and the number of distinct hyperplanes incident to β. This is no longer the case if β has codimension greater than two.

5 Examples and Computations

In dimension $d > 2$ the homology modules can be quite complicated, in contrast to the planar case. For $d = 2$, the only module of interest is $H_0(\mathcal{J})$, which has a simple presentation (see [18]). In higher dimensions, we must also determine kernels of maps, and the presentations become correspondingly more complex. The following examples were computed using the Macaulay II package ([13]).

We begin with an octahedron, triangulated by placing a single vertex in the interior. For a central configuration such as this, it is easy to see that $H_0(\mathcal{J})$ vanishes, so we want to study the module $H_1(\mathcal{J})$. For this example, there can be anywhere from three to twelve distinct hyperplanes incident to the central vertex.

Example 5.1 Let Δ correspond to the regular octahedron, triangulated by placing a centrally symmetric vertex.

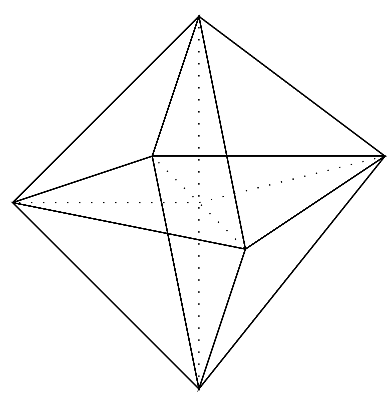


$H_1(\mathcal{J})$ is always zero for this configuration (this can be worked out by hand, but it is tedious), so $C^r(\hat{\Delta})$ is always free. If we apply Corollary 4.12, we find that the numerator of the Hilbert series is

$$P(H_3(\mathcal{R}/\mathcal{J}), t) = \sum_{i=0}^{3} (-1)^i P(\bigoplus_{\beta \in \Delta^0_{3-i}} \mathcal{R}/\mathcal{J}(\beta), t).$$

This equals $8 - 12(1 - t^{r+1}) + 6(1 - 2t^{r+1} + t^{2r+2}) - (1 - 3t^{r+1} + 3t^{2r+2} - t^{3r+3})$, which simplifies to $1 + 3t^{r+1} + 3t^{2r+2} + t^{3r+3}$. Since $C^r(\hat{\Delta})$ is free, we can read off the number of generators in each degree; with a few moments of thought, it is easy to visualize exactly what these generators must be. This configuration is highly non-generic; we have three sets of five coplanar vertices.

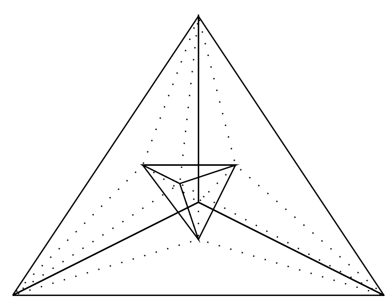
Example 5.2 By pulling one of the boundary vertices along one of the four boundary edges incident to it (a non-generic perturbation), we obtain an example with four hyperplanes incident to the center vertex:



Computations show that we again have $H_1(\mathcal{J}) = 0$ for all r. Finally, if we drag the boundary vertex nearest the reader along the boundary edge connecting it to the bottom vertex, we obtain an example with six hyperplanes incident to the center vertex, but in which we have two sets of five coplanar vertices. This example also has $C^r(\hat{\Delta})$ free for all r.

These cases are all non-generic, and they seem to be the exception. In particular, if we consider a generic case (i.e. no set of four vertices of the octahedron is coplanar), then $H_1(\mathcal{J})$ does vanish if $r = 1$, but is nonzero for all $r > 1$.

Example 5.3 Let Δ correspond to a tetrahedron, triangulated by placing a smaller, inverted tetrahedron totally inside it (i.e. the interior tetrahedron does not meet the boundary of the larger tetrahedron), and then connecting vertices as in the diagram below. For this example, we have fifteen top dimensional simplices, twenty-eight interior two faces, eighteen interior edges, and four interior vertices.



We again begin with a highly non-generic configuration, the three-dimensional analog of the Morgan-Scott [15] example. Place the vertices so that if we connect each boundary vertex to the (unique) interior vertex with which it does not share a common edge, then the four lines obtained this way all meet in a point. For this case, $H_0(\mathcal{J})$ is zero for $r = 1$, but $H_1(\mathcal{J})$ has associated primes of codimension three and four. For a generic configuration of the same combinatorial type, we find that in the $r = 1$ case, $H_1(\mathcal{J})$ has only a codimension four associated prime.

In [18], we showed that in the planar case, $C^1(\hat{\Delta})$ is generically free (this result follows from the work of Billera and Whiteley in [4], [21]). The previous example shows that this result does not hold in higher dimensions. However, in the generic case, for $r = 1$, $H_1(\mathcal{J})$ seems to have only the maximal ideal as an associated prime. Based on this and other three dimensional examples, a possible generalization of the result for the planar case to higher dimensions is the following:

Conjecture 5.4 *If Δ is a topological d-ball, triangulated generically, and $r = 1$, then $H_i(\mathcal{J})$ is supported on primes of codimension greater than $d - i + 1$, for $i < d - 1$.*

For example, this would imply that for generic simplicial complexes with $d = 3, r = 1$, the number of splines in sufficiently high degree is determined by local data. Alfeld, Schumaker, and Whiteley [3] recently proved that this is indeed the case.

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