A family of ideals of minimal regularity and the Hilbert series of $C^r(\hat{\Delta})$

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Hilbert series of $C^r(\hat{\Delta})$

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Abstract

For a simplicial subdivison Δ of a region in $\mathbf{R}^2$, we analyze the dimension of the vector space $C^r_k(\Delta)$ of C^r piecewise polynomial functions (splines) on Δ of degree at most k. We find an exact sequence which allows us to prove that the dimension series for splines given by Billera and Rose in [5] does indeed agree with the bounds on the dimension of the spline space given by Alfeld and Schumaker in [1], [2]. We give sufficient conditions for the Alfeld-Schumaker bounds to be attained in all degrees, where Δ is a two-dimensional simplicial complex. The conditions are satisfied by the class of complexes considered by Chui and Wang in [6], but also by a much broader class of complexes. Furthermore, for conditions which involve only local geometric data, this result is the strongest possible.

1 Introduction

The dimension of the space of r-differentiable piecewise polynomial functions over a polyhedral subdivision is a much studied topic in approximation theory. Given a subdivision of a pseudomanifold Δ in $\mathbf{R}^2$, we denote as $C^r_k(\Delta)$ the vector space of C^r functions on Δ, which are given locally (i.e. on each two cell of Δ) by a polynomial of degree less than or equal to k. By embedding $\mathbf{R}^2$ in $\mathbf{R}^3$ and forming the cone $\hat{\Delta}$ of Δ with the origin in $\mathbf{R}^3$, we turn the problem of computing the dimension of $C^r_k(\Delta)$ into a question about graded modules and graded maps, and may apply the tools of commutative and homological algebra to the problem. For example, the dimension of $C^r_k(\Delta)$ is exactly equal to the dimension of $C^r(\hat{\Delta})_k$. This approach first appeared in Billera and Rose [5]. In this paper, we restrict our attention to simplicial subdivisions. With this restriction, the dimension of the spaces $C^r_k(\Delta)$ is known for certain triangulations, but not in general.

The dimension of the space $C^r_k(\Delta)$ is determined by three different aspects of Δ: the combinatorics, the local geometry (the number of lines of distinct slope incident to a given interior vertex), and the global geometry (which is measured, in some sense, by local cohomology). In [8], we studied the connection between the local cohomology and Δ. In this paper, we consider situations in which the local cohomology vanishes, and show that in these situations, the module $C^r(\hat{\Delta})$ (and hence the Hilbert series and the dimension of $C^r_k(\Delta)$, for all k) is completely determined by the combinatorial and local geometric data. The main theorem presented here (Theorem 5.2) gives sufficient conditions for the vanishing of the local cohomology.

The organization of the paper is as follows: in §2 we set up our notation and review the earlier results we will need. In §3, we explore the free resolution of $\mathbf{R}$[x,y]$/J$, where J is an ideal of homogeneous linear forms, each raised to a fixed power. In §4, we use this and the results of §2 to obtain results on the Hilbert series of $C^r(\hat{\Delta})$. In §5, we prove the aforementioned theorem on the vanishing of local cohomology, and give some examples.

The examples which appear in this paper (and many others) were computed using the Macaulay II package, written by Mike Stillman and Dan Grayson. The package includes a script (spline) which takes an abstract simplicial complex, embedding information, and the desired order of smoothness, and builds a chain complex, such that $C^r(\hat{\Delta})$ is the top homology module of the complex. Then another script (homology) allows one to compute the homology modules of any order.

2 Preliminaries

Let Δ be a connected finite simplicial complex which is supported on $|\Delta| \subset \mathbf{R}^2$. Let $r \geq 0$ be an integer, and let $R = \mathbf{R}[x, y, z]$. Define the spaces of splines:

$C^r_k(\Delta) = \{F : |\Delta| \to \mathbf{R} : F|_\sigma$ is a polynomial of degree $\leq k$, for all $\sigma \in \Delta_2$, and F is continuously differentiable of order r$\}$

$C^r(\hat{\Delta}) = \{F : |\hat{\Delta}| \to \mathbf{R} : F|_{\hat{\sigma}} \in R$, for all $\sigma \in \Delta_2$, and F is continuously differentiable of order r$\}$

where $\hat{\Delta}$ is the join of Δ with the origin in $\mathbf{R}^3$. $C^r(\hat{\Delta})$ is a finitely generated graded R-module.

Throughout this paper, Δ denotes a connected finite simplicial complex, such that Δ and all its links are pseudomanifolds. Δ_i, Δ_i^0 denote (respectively) the sets of i-dimensional faces and i-dimensional interior faces. f_i and f_i^0 denote the cardinality of the preceding sets. For $\epsilon \in \Delta_1^0$, let l_ϵ be the homogeneous linear form vanishing on $\hat{\epsilon}$, and for $v_i \in \Delta_0^0$, define $J(v_i) = \sum_{v_i \in \epsilon} l_\epsilon^{r+1}$, i.e. $J(v_i)$ is the ideal generated by the $r+1^{st}$ powers of the linear forms vanishing on edges terminating in v_i.

Lemma 2.1 *(Billera and Rose [5]) There is a (graded) exact sequence:*

$$0 \longrightarrow ker(\phi) \longrightarrow R^{f_2} \bigoplus R^{f_1^0}(-r-1) \xrightarrow{\phi} R^{f_1^0} \longrightarrow coker(\phi) \longrightarrow 0$$

$$where\ \phi = \left(\begin{array}{c|ccc} & l_{\epsilon_1}^{r+1} & & \\ \partial_2 & & \ddots & \\ & & & l_{\epsilon_{f_1^0}}^{r+1} \end{array} \right).$$

∂_2 is the simplicial (reduced) boundary map from $R^{f_2} \longrightarrow R^{f_1^0}$. Billera and Rose showed that $C^r(\hat{\Delta})$ is isomorphic to the kernel of ϕ. As it will appear often in the remainder of the paper, we define $N = \text{coker}(\phi)$. By analyzing N, we may obtain information about $C^r(\hat{\Delta})$. We now summarize the results from [8] which we will use.

Lemma 2.2 *The module N is related to the local geometry of Δ (i.e. the module $\bigoplus_{v \in \Delta_0^0} R/J(v)$) by the following (graded) exact sequence:*

$$0 \longrightarrow H_m^0(N) \longrightarrow N \longrightarrow \bigoplus_{v \in \Delta_0^0} R/J(v) \longrightarrow 0$$

$H^0_m(N)$ is the zeroth local cohomology of N at the ideal $m = <x, y, z>$, so $H^0_m(N) = \{a \in N \mid <x, y, z>^n \cdot a = 0$, some $n \gg 0\}$. $H^0_m(N)$ is a graded module over R, and vanishes in sufficiently large degree.

If $H^0_m(N)$ vanishes, then the module N is isomorphic to the module $\bigoplus_{v \in \Delta^0_0} R/J(v)$, and the study of N reduces to the study of the individual elements appearing in the direct sum. In [8], we show that if Δ has positive genus (where we use genus to denote the rank of the first reduced simplicial homology module), then $H^0_m(N)$ is nonzero. The results in §3 and §4 hold in general, but in §5 we restrict to the case where Δ has genus zero, i.e. Δ is a (topological) disk. In this case, there is a simple presentation for $H^0_m(N)$.

Lemma 2.3 *Let Δ be a disk in the plane, and define $K^r \subset \bigoplus_{\epsilon_i \in \Delta^0_1} R\epsilon_i$ to be the submodule generated by*

$$\{\epsilon_i \mid \epsilon_i \text{ not totally interior}\},$$

(an edge is totally interior if both its vertices are interior), and for each interior vertex v the set

$$\{\sum_{v \in \epsilon_i} a_{\epsilon_i} \epsilon_i \mid \text{there exists a relation } \sum_{v \in \epsilon_i} a_{\epsilon_i} l^{r+1}_{\epsilon_i} = 0, \text{for } a_{\epsilon_i} \in R\}.$$

Then $H^0_m(N)$ is given by generators and relations by:

$$0 \longrightarrow K^r \longrightarrow \bigoplus_{\epsilon_i \in \Delta^0_1} R\epsilon_i \longrightarrow H^0_m(N) \longrightarrow 0.$$

In [8], we obtained a result which showed why the examples considered by Chui and Wang always had $H^0_m(N) = 0$. Define a relation on interior edges by $\epsilon_i \sim \epsilon_j$ if ϵ_i and ϵ_j share a vertex, and have the same affine hull. Taking the transitive closure of this relation gives us an equivalence relation on interior edges. We call ϵ_i a pseudoboundary edge if $\epsilon_i \sim \epsilon_j$ with ϵ_j not totally interior.

Definition 2.4 *For each edge τ of Δ, define*

$$s_\tau := \max\{\text{number of slopes at } v_i,\ v_i \text{ a vertex of an edge } \sigma,\ \sigma \sim \tau\}.$$

Define an integer $s(\Delta)$ by

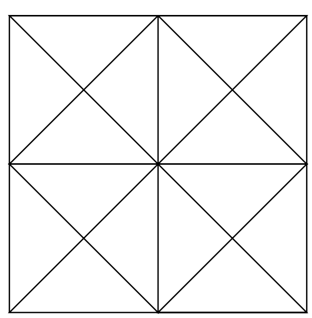
$$s(\Delta) := \min\{s_\tau \mid \tau \text{ an edge which is not a pseudoboundary edge of } \Delta\}.$$

Lemma 2.5 *If Δ is a triangulation of a disk in the plane, then*

(a) If every edge of Δ is a pseudoboundary edge, then $H^0_m(N) = 0$ for all r.

(b) If Δ has at least one edge which is not a pseudoboundary, then for each $r \geq s(\Delta) - 2$, $H^0_m(N) \neq 0$.

Example 2.6 An example of a triangulation where every edge is a pseudoboundary. Lemma 2.5 implies $H^0_m(N) = 0$ for all r.



The results of §4 will enable us to write down the Hilbert series of the module of splines on this configuration, for any r.

3 The ideal $J(v)$

Let $R = \mathbf{R}[x, y, z]$, and let Δ be a finite simplicial complex embedded in the plane. Since we are considering $J(v)$ for a specific v, we may translate v to the point $(0, 0, 1)$ in $\mathbf{R}^3$. Then the linear forms whose powers define $J(v)$ may be written as $x + a_i y$. So we may assume that $J(v)$ has the form $< (x + a_1 y)^{r+1}, \cdots, (x + a_k y)^{r+1} >$, with $a_i \neq a_j$ if $i \neq j$. For ease of notation, define $\alpha(v) = \lfloor \frac{r+1}{k-1} \rfloor$.

Theorem 3.1 *A free resolution of $R/J(v)$ is given by:*

$$R(-r-1-\alpha(v))^{s_1} \bigoplus R(-r-2-\alpha(v))^{s_2} \to R(-r-1)^k \to R \to R/J(v) \to 0,$$

where $s_1 = (k-1)\alpha(v) + k - r - 2$ and $s_2 = r + 1 - (k-1)\alpha(v)$

Proof. Since $J(v)$ is an ideal in two variables, we may consider it as an ideal in $\mathbf{R}[x, y]$, so by the Hilbert Syzygy Theorem (see [7]), the projective dimension of $R/J(v) \leq 2$. But since there are at least two distinct lines at each vertex, the projective dimension of $R/J(v) = 2$, and the Hilbert-Burch theorem (again, see [7]) tells us that $J(v)$ is generated by the $k-1$ by $k-1$ minors of a k by $k-1$ matrix whose entries are homogeneous polynomials in $\mathbf{R}[x, y]$. In particular, the columns of the matrix are the syzygies on $J(v)$, so we get a rough bound on the possible degrees of the syzygies. Since the generators of $J(v)$ are homogeneous of the same degree, any syzygy is homogeneous. We want to find the minimal generators of the syzygy module. Define an $r+d+2$ by $d+1$ matrix

$$D(i,r,d) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \binom{r+1}{1}a_i & 1 & \ddots & \vdots \\ \binom{r+1}{2}a_i^2 & \binom{r+1}{1}a_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ \binom{r+1}{r+1}a_i^{r+1} & \binom{r+1}{r}a_i^r & \ddots & \vdots \\ 0 & \binom{r+1}{r+1}a_i^{r+1} & \ddots & \vdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \binom{r+1}{r+1}a_i^{r+1} \end{pmatrix}$$

Now define an $r+d+2$ by $k(d+1)$ matrix $M(r,k,d)$

$$M(r,k,d) = (\, D(1,r,d) \quad D(2,r,d) \quad \cdots \quad D(k,r,d)\,)$$

Let $s = (s_1, \cdots, s_k)$ be a syzygy of degree d; say $s_i = b_{i0}x^d + b_{i1}x^{d-1}y + \cdots + b_{id}y^d$. Let $\mathbf{b} = (b_{10}, \cdots, b_{1d}, b_{20}, \cdots, b_{2d}, b_{k0}, \cdots, b_{kd})$. $\sum_{i=1}^k s_i l_i^{r+1} = 0$ if and only if $\mathbf{b}$ is in the kernel of $M(r,k,d)$. In Lemma 3.2, we prove that $M(r,k,d)$ has full rank, so that $\mathrm{nullity}(M(r,k,d)) = k(d+1) - (r+d+2)$.

If $d < \alpha(v)$, then since $k(d+1)-(r+d+2) \leq 0 \iff d \leq \frac{r+2-k}{k-1} = \frac{r+1}{k-1} - 1 < \alpha(v)$, we see that $\ker(M(r,k,d)) = 0$, so there are no syzygies of degree $< \alpha(v)$.

If $d = \alpha(v)$, then since $\alpha(v)$ is the smallest integer greater than $\frac{r+1}{k-1} - 1$, from $k(d+1)-(r+d+2) > 0 \iff d > \frac{r+2-k}{k-1} = \frac{r+1}{k-1} - 1$, we see that $\ker(M(r,k,\alpha(v))) \neq 0$, with $\mathrm{nullity}(M(r,k,\alpha(v))) = s_1$.

Each syzygy of degree $\alpha(v)$ lifts to give two syzygies of degree $\alpha(v)+1$. Since $\text{nullity}(M(r,k,\alpha(v)+1)) = \text{nullity}(M(r,k,\alpha(v))) + k - 1$, if we assume that the lifted syzygies are all independent, then there must be $s_2 = \text{nullity}(M(r,k,\alpha(v)+1)) - 2s_1$ syzygies of degree $\alpha(v)+1$ which are not lifted from syzygies of degree $\alpha(v)$. Checking, we find that $\alpha(v)s_1 + (\alpha(v)+1)s_2 = r+1$, and $s_1 + s_2 = k-1$, so indeed the lifted syzygies are independent, and the Hilbert-Burch theorem tells us that we have found all the syzygies. $\square$

Lemma 3.2 *(Schumaker) The matrix $M(r,k,d)$ has full rank, i.e. rank $M(r,k,d) = max(r+d+2, k(d+1))$.*

Proof. Step 1: Modify the matrix $D(i,r,d)$ via elementary column operations to obtain the matrix:

$$D'(i,r,d) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \binom{r+d+1}{1}a_i & 1 & \ddots & \vdots \\ \binom{r+d+1}{2}a_i^2 & \binom{r+d}{1}a_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & \ddots & \binom{r+1}{1}a_i \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \binom{r+d+1}{r+d+1}a_i^{r+d+1} & \binom{r+d}{r+d}a_i^{r+d} & \cdots & \binom{r+1}{r+1}a_i^{r+1} \end{pmatrix}$$

To see that this is possible, regard the columns as representations of univariate polynomials with respect to the ordered basis $t^{r+d+1}, t^{r+d}, \ldots, 1$. So then the last two columns of $D(i,r,d)$ represent the polynomials $t(t+a_i)^{r+1}$ and $(t+a_i)^{r+1}$, respectively.

Adding a_i times the last column to the next to last column yields a column representing the polynomial $(t+a_i)^{r+2}$, which is what we have written down in $D'(i,r,d)$. Repeat.

Step 2: We now have a matrix $D'(i,r,d)$ whose columns represent the polynomials $(t+a_i)^{r+d+1},\ldots,(t+a_i)^{r+1}$ with respect to the above ordered basis. Scale the columns by (non-zero) constants so that (still viewing the columns as univariate polynomials) the j^{th} column is the derivitive of the $(j-1)^{st}$ column.

Step 3: Define

$$M'(r,k,d) = \begin{pmatrix} D'(1,r,d) & D'(2,r,d) & \cdots & D'(k,r,d) \end{pmatrix}$$

Let $\mathbf{c} = (c_0,\ldots,c_{r+d+1})$, and consider the solutions to $\mathbf{c}\cdot M'(r,k,d) = 0$. View $\mathbf{c}$ as the polynomial $p = \sum_{i=0}^{r+d+1} c_i \binom{r+d+1}{i} y^i$. Then $\mathbf{c}\cdot D'(i,r,d) = 0$ if and only if $p(a_i) = \frac{dp(a_i)}{dy} = \ldots = \frac{dp^{(d)}(a_i)}{dy} = 0$, i.e. if and only if p has a root of order $d+1$ at a_i, which occurs if and only if $(t-a_i)^{d+1}$ divides p. So $\mathbf{c}$ is in the (left) kernel of $M'(r,k,d)$ if and only if $\prod_{i=1}^{k}(t-a_i)^{d+1}$ divides p. For $(r+d+2) > k(d+1)$, there are $(r+d+2)-k(d+1)$ such polynomials, and for $(r+d+2) \leq k(d+1)$, there are none. So rank $M'(r,k,d) =$ rank $M'(r,k,d)^t = k(d+1)$ if $r+d+2 > k(d+1)$, and $r+d+2$ if $r+d+2 \leq k(d+1)$, i.e. $M'(r,k,d)$ has full rank. $\square$

Definition 3.3 *A graded R-module M is n-regular if there exists a free resolution*

$$0 \to \oplus_j R(-e_{rj}) \to \ldots \to \oplus_j R(-e_{1j}) \to \oplus_j R(-e_{0j}) \to M \to 0$$

of M, with $e_{ij} - i \leq n$ for all i,j.

Corollary 3.4 *The regularity of $R/J(v)$ is $r + \lceil \frac{r+1}{k-1} \rceil - 1$.*

Proof. $s_2 = 0 \iff r + 1 - (k-1)\alpha(v) = 0 \iff \alpha(v) = \frac{r+1}{k-1} \iff \frac{r+1}{k-1} \in \mathbf{Z}$. So the regularity of $R/J(v)$ is $r - 1 + \alpha(v)$ if $\frac{r+1}{k-1} \in \mathbf{Z}$, and $r + \alpha(v)$ otherwise. But if $\frac{r+1}{k-1} \in \mathbf{Z}$, then $\alpha(v) = \lceil \frac{r+1}{k-1} \rceil$, and if $\frac{r+1}{k-1} \notin \mathbf{Z}$, then $\alpha(v) = \lceil \frac{r+1}{k-1} \rceil - 1$. $\square$

Since the degrees of the $k-1$ syzygies on $J(v)$ must sum to $r+1$, the minimal possible regularity for $J(v)$ arises from dividing $r+1$ as "evenly" as possible by $k-1$, which is indeed the case here, i.e. for an ideal of projective dimension two, generated in degree $r+1$, $J(v)$ has the minimal possible regularity.

4 The Hilbert Series of $C^r(\hat{\Delta})$

If N is a graded module over $R = \mathbf{R}[x, y, z]$, the Hilbert series of N is the formal power series $\sum_{d \geq 0} dim(N_d)t^d$. The Hilbert series has a generating function of the form $P(N,t)/(1-t)^3$, where $P(N,t) \in \mathbf{Z}[t]$ (for more details on the Hilbert series, see [5] or [7]). The Hilbert series behaves additively on exact sequences, so from the exact sequence:

$$0 \longrightarrow C^r(\hat{\Delta}) \longrightarrow R^{f_2} \bigoplus R^{f_1^0}(-r-1) \longrightarrow R^{f_1^0} \longrightarrow N \longrightarrow 0,$$

we obtain

$$P(C^r(\hat{\Delta}), t) = P(R^{f_2}, t) + P(R^{f_1^0}(-r-1), t) - P(R^{f_1^0}, t) + P(N, t),$$

so

$$P(C^r(\hat{\Delta}), t) = f_2 + f_1^0 t^{r+1} - f_1^0 + P(N, t).$$

Thus, the problem is to determine $P(N,t)$.

Lemma 4.1

$$P(R/J(v_i),t) = 1 - k(v_i)t^{r+1} + s_1(v_i)t^{r+1+\alpha(v_i)} + s_2(v_i)t^{r+2+\alpha(v_i)}$$

where $\alpha(v_i), k(v_i), s_1(v_i), s_2(v_i)$ *are the values of the* $\alpha(v), k, s_1, s_2$ *appearing in Theorem 3.1, for the vertex* v_i.

Proof. Immediate from Theorem 3.1. □

From the short exact sequence of Lemma 2.2, we have:

$$P(N,t) = P(H^0_m(N),t) + \sum_{v_i \in \Delta^0_0} P(R/J(v_i),t),$$

which combined with Lemma 4.1 proves:

Corollary 4.2 *If* $H^0_m(N) = 0$, *then*

$$P(N,t) = \sum_{v_i \in \Delta^0_0} 1 - k(v_i)t^{r+1} + s_1(v_i)t^{r+1+\alpha(v_i)} + s_2(v_i)t^{r+2+\alpha(v_i)}.$$

Corollary 4.3 *(Billera and Rose, [5])*

$P(C^r(\hat{\Delta}),1) = f_2$

$P'(C^r(\hat{\Delta}),1) = (r+1)f^0_1$

$P(C^r(\hat{\Delta}),0) = 1$

Proof. Since $H^0_m(N)$ vanishes in sufficiently high degree, $P(H^0_m(N),t) = (1-t)^3 g(t)$. Degree zero generators of $H^0_m(N)$ correspond to generators of the simplicial (reduced) homology of Δ (see [8]), so $g(0)$ is equal to the genus g of Δ.

Evaluating $P(H^0_m(N),t)$, we find $P(H^0_m(N),1) = 0 = P'(H^0_m(N),1)$, while $P(H^0_m(N),0) = g$.

Evaluating $P(R/J(v_i),t)$, we find $P(R/J(v_i),1) = P'(R/J(v_i),1) = 0$, and $P(R/J(v_i),0) = 1$.

Applying Lemma 2.2, we find that $P(N,1) = P'(N,1) = 0$, and $P(N,0) = f_0^0 + g$. The first two equalities of the corollary are immediate, and the last follows since $f_2 - f_1^0 + f_0^0 + g = 1$, which is easily derived from Eulers formula. □

Corollary 4.4 *For $r = 1$, if $H^0_m(N)$ vanishes and s denotes the number of "singular" vertices (i.e., vertices with $k(v_i) = 2$), then*

$$P(C^1(\hat{\Delta}),t) = 1 + (s - 3f_0^0 + f_1^0)t^2 + 2(f_0^0 - s)t^3 + st^4.$$

Proof. From Lemma 4.1, we have:

$P(R/J(v_i),t) = 1 - 2t^2 + t^4$ if $k(v_i) = 2$

$P(R/J(v_i),t) = 1 - 3t^2 + 2t^3$ if $k(v_i) \geq 3$

Summing over all $v_i \in \Delta_0^0$ yields

$$\sum_{v_i \in \Delta_0^0} P(R/J(v_i),t) = f_0^0 + (s - 3f_0^0)t^2 + 2(f_0^0 - s)t^3 + st^4.$$

Now apply Corollary 4.2 and the formula preceding Lemma 4.1. □

We can now prove that the homological approach used by Billera and Rose to determine the dimension of the spline space agrees with the Bezier-Bernstein polynomial approach of Alfeld and Schumaker.

Corollary 4.5 *(Alfeld and Schumaker)*

If $k \gg 0$, then:

$$\dim C^r_k(\Delta) = \binom{k+2}{2} + \binom{k-r+1}{2} f^1_0 - \left(\binom{k+2}{2} - \binom{r+2}{2}\right) f^0_0 + \sigma,$$

where $\sigma = \sum \sigma_i$ and $\sigma_i = \sum_j (r+1+j(1-k(v_i))_+$. In fact, Alfeld and Schumaker prove that this bound is attained for $k \geq 3r+1$.

Proof. Since the dimension of $C^r_k(\Delta)$ is the k^{th} coefficient in the Hilbert series of $C^r(\hat{\Delta})$, and since $H^0_m(N)$ vanishes in sufficiently high degree, we just plug the formula of Corollary 4.2 into the equation preceding Lemma 4.1, and then compute the coefficients. The only non-obvious part comes from the equality:

$$\sigma_i = \frac{1}{2}((1-k(v_i))\alpha(v_i)^2 + (2r - k(v_i) + 3)\alpha(v_i)),$$

which is easy to prove. Notice that the (well-known) conjecture $\dim C^1_3 = 3f^\partial_0 + 2f^0_0 + 1 + s$ is equivalent to showing that in the $r = 1$ case, $H^0_m(N)$ vanishes in degree greater than two (The Alfeld-Schumaker result proves that for a given r, $H^0_m(N)$ vanishes in degree greater than $3r$). □

5 The vanishing of $H^0_m(N)$ and some examples

By Corollary 4.2 and the equation preceding Lemma 4.1, if $H^0_m(N) = 0$, then the Hilbert series (and hence also $C^r_k(\Delta)$) is entirely determined by local geometric data. In this situation, we can actually write down the Hilbert series by observation. In [10], Whiteley showed that if $r = 1$, then $H^0_m(N)$ vanishes generically. However, the generic vanishing depends on *global* data, and in general it is difficult (i.e. computationally expensive) to determine if one is really in a generic situation. A configuration for which $H^0_m(N) = 0$ for all r is given by Lemma 2.5: the configuration in which every interior edge is a pseudoboundary. Chui and Wang [6] considered this situation, though in language different from that of this paper. The point is that there are many configurations in which not every interior edge is a pseudoboundary, but where the local geometric data is sufficient to prove that $H^0_m(N)$ must be zero. From the presentation of Lemma 2.3, $H^0_m(N) = 0$ if every totally interior edge is connected by a degree zero syzygy to other interior edges which, when considered as generators of $H^0_m(N)$, are all zero.

Definition 5.1 *For $v \in \Delta_0$, define $d(v, \partial)$ to be the smallest number of edges in a path connecting v to a vertex in the boundary. For $\epsilon \in \Delta_1$, let $\partial(\epsilon) = v_i - v_j$, and define $d(\epsilon, \partial) = \inf\{d(v_i, \partial), d(v_j, \partial)\}$.*

Theorem 5.2 *Let Δ be a simplicial complex supported on a (topological) disk. Suppose there exists a total order $\prec$ on Δ^0_1 which satisfies:*

1. *$\tau \prec \epsilon$ if $d(\tau, \partial) = 0$ and $d(\epsilon, \partial) > 0$.*
2. *For each totally interior edge ϵ, either (a) or (b) holds*
   - (a) *There exist edges $\tau_1, \ldots, \tau_{r+2} \prec \epsilon$, such that $\tau_1, \ldots, \tau_{r+2}, \epsilon$ share a common vertex, and such that $l_{\tau_1}, \ldots, l_{\tau_{r+2}}$ have distinct slopes.*

(b) There exists an edge $\epsilon' \prec \epsilon$ which shares a vertex with ϵ, and such that $l_{\epsilon'} = l_\epsilon$.

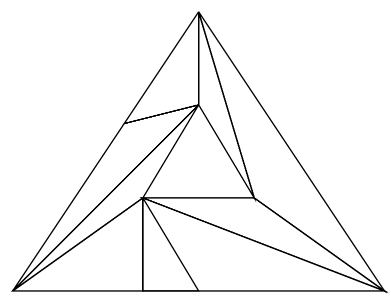
Then $H^0_m(N) = 0$.

Proof. Recall the presentation for $H^0_m(N)$ given in Lemma 2.3. We need to annihilate the generators of $H^0_m(N)$, which are the totally interior edges. Order the interior edges via $\prec$, and let ϵ be the first totally interior edge in the order. In case(a), there is a degree zero syzygy on $(l_{\tau_1}^{r+1}, \ldots, l_{\tau_{r+2}}^{r+1}, l_\epsilon^{r+1})$, where $\tau_1, \ldots, \tau_{r+2}$ are edges which are not totally interior (and hence which are zero in $H^0_m(N)$). In case(b), there is a degree zero syzygy on $(l_{\epsilon'}^{r+1}, l_\epsilon^{r+1})$, where ϵ' is zero in $H^0_m(N)$. So in both cases, the generator ϵ vanishes in $H^0_m(N)$. Now repeat the process. □

We return to the configuration Δ given in Example 2.6. Since $H^0_m(N) = 0$, we may compute the Hilbert series for the spline module by considering the number of lines of distinct slope at the interior vertices. Δ has four interior vertices with $k = 2$, and one interior vertex with $k = 4$. If we let $r = 1$, then for the $k = 2$ case, $P(\mathcal{R}/\mathcal{J}(v), t) = 1 - 2t^2 + t^4$; for the $k = 4$ case $P(\mathcal{R}/\mathcal{J}(v), t) = 1 - 3t^2 + 2t^3$. So $P(N, t) = 5 - 11t^2 + 2t^3 + 4t^4$, and $P(C^1(\hat{\Delta}), t) = 1 + 9t^2 + 2t^3 + 4t^4$. In [8], we prove that if $H^0_m(N) = 0$, then $C^r(\hat{\Delta})$ is free; in this event we can read off the number of generators and their degrees from $P(C^r(\hat{\Delta}), t)$.

Theorem 5.2 allows us to use *local* geometric data to force the vanishing of $H^0_m(N)$, essentially by "peeling the onion". That is, for a given vertex v at distance one from the boundary, if there are the right kind of edges connecting v to the boundary, then we can kill off the interior edges adjacent to v, then repeat the process. One consequence of this is that for a given configuration, it is always possible to subdivide and obtain a subdivision where $H^0_m(N)$ vanishes. We conclude with some additional examples.

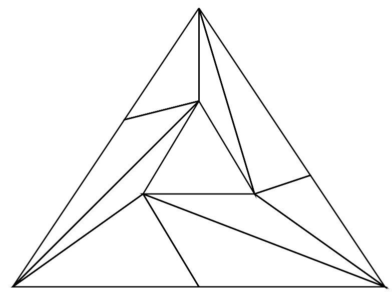
Example 5.3 Let $r = 2$, and let Δ correspond to the following complex.



By Theorem 5.2, this configuration has $H^0_m(N) = 0$, so we may apply Corollary 4.2 to find $P(N, t)$. At one interior vertex we have $k = 4, s_1 = 3, s_2 = 0$; at the second interior vertex we have $k = 5, s_1 = 1, s_2 = 3$; and at the remaining interior vertex we have $k = 6, s_1 = 2, s_2 = 3$. So $P(N, t) = (1 - 4t^3 + 3t^4) + (1 - 5t^3 + t^3 + 3t^4) + (1 - 6t^3 + 2t^3 + 3t^4) = 3 - 12t^3 + 9t^4$. Since $f_2 = 10$ and $f^0_1 = 12$, we obtain $P(C^2(\hat{\Delta}), t) = 1 + 9t^4$. If we let $r = 3$, then by Lemma 2.5, $H^0_m(N) \neq 0$, so we cannot rely on local data to compute $P(C^3(\hat{\Delta}), t)$.

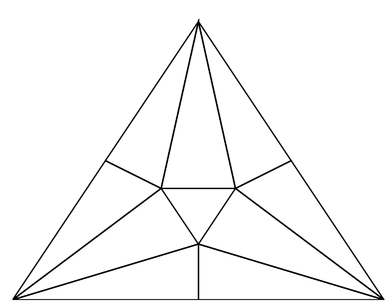
Theorem 5.2 is as strong as possible, in the following sense: if only local geometric data is considered, then the only way to force $H^0_m(N)$ to vanish is via the conditions of Theorem 5.2. We now give a pair of examples, whose local geometric data coincide, but where the local geometric data do not force $H^0_m(N)$ to vanish.

Example 5.4 Again, let $r = 2$, and let Δ correspond to the complex:



For this configuration, $H^0_m(N)$ is generically zero. One way to see this is to compute a resolution for N (Notice that this entails a specific choice of vertex positions). Since the resolution has length two, the Auslander-Buchsbaum formula [7] implies that $depth(N) = 1$, and since $H^i_m(N) = 0$ if $i < depth(N)$ (again, see [7]), $H^0_m(N) = 0$. The three totally interior vertices all have $J(v)$ consisting of 5 different lines, so $k = 5, s_1 = 1, s_2 = 3$, for each interior vertex. Applying Corollary 4.2, we have $P(N,t) = 3(1-5t^3+t^3+3t^4)$. Finally, $f_2 = 10$, and $f^0_1 = 12$, so $P(C^2(\hat{\Delta}),t) = 1+9t^4$.

Example 5.5 Locally, this configuration appears the same as the previous example; that is, at each interior vertex we have $k = 5$, and corresponding values for s_1, s_2. But $H^0_m(N) \neq 0$ for this configuration. (The defining property of this configuration is that the three interior edges which are orthogonal to the boundary have affine hulls which contain two boundary vertices and one interior vertex, and the hulls meet at a point)



For this example, $H^0_m(N)$ is a one dimensional vector space, appearing in degree 3, so $P(H^0_m(N),t) = t^3(1-t)^3$. Thus, by the equation preceding Corollary 4.3, $P(N,t) = t^3(1-t)^3 + 3(1-5t^3+t^3+3t^4)$ and $P(C^2(\hat{\Delta}),t) = 1+t^3+6t^4+3t^5-t^6$.

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