A rank two vector bundle associated to a three arrangement, and its Chern polynomial

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Abstract

We prove that the Poincare polynomial $\pi(\mathcal{A}, t)$ of a (central) three arrangement $\mathcal{A}$ is $(1+t) \cdot c_t(\mathcal{D}_0(\mathcal{A})^\vee)$, where $\mathcal{D}_0(\mathcal{A})$ is a direct summand of the sheaf associated to the Terao module $D(\mathcal{A})$ of $\mathcal{A}$, and c_t is the Chern polynomial. We also prove that if $\mathcal{A}$ is a (central) three arrangement, then $\mathcal{D}_0(\mathcal{A})^\vee$ is a vector bundle on $\mathbf{P}^2$, and derive an algorithm which computes $c_t(\mathcal{D}_0(\mathcal{A})^\vee)$ from a resolution of the Jacobian of the defining polynomial of $\mathcal{A}$.

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1 Introduction

A hyperplane arrangement $\mathcal{A}$ is a finite collection of codimension one linear subspaces of a fixed vector space $\mathbf{V}$. $\mathcal{A}$ is *central* if each hyperplane contains the origin $\mathbf{0}$ of $\mathbf{V}$. Arrangements can be studied from a wide variety of viewpoints; combinatorics, topology, algebra, and geometry all play important roles; for more details see Orlik [5] or Orlik and Terao [6]. A fundamental invariant of $\mathcal{A}$ is the Poincare polynomial $\pi(\mathcal{A}, t)$. There are various ways of defining $\pi(\mathcal{A}, t)$; the simplest is from the intersection lattice $L_{\mathcal{A}}$ of $\mathcal{A}$. $L_{\mathcal{A}}$ consists of the intersections of the elements of $\mathcal{A}$, ordered by reverse inclusion. $\mathbf{V}$ is the lattice element $\hat{0}$; the rank one elements are the hyperplanes themselves.

Definition 1.1 *The Möbius function* $\mu : L_{\mathcal{A}} \longrightarrow \mathbf{Z}$ *is defined by*

$$\begin{aligned} \mu(\hat{0}) &= 1 \\ \mu(t) &= -\sum_{s<t} \mu(s), \text{ if } \hat{0} < t \end{aligned}$$

Definition 1.2 *The Poincare polynomial* $\pi(\mathcal{A}, t)$ *is defined by*

$$\pi(\mathcal{A}, t) = \sum_{X \in L_{\mathcal{A}}} \mu(X) \cdot (-t)^{rank(X)}$$

Example 1.3 Let $\mathcal{A}$ be the arrangement defined by the vanishing of $x, y, z, x+y+z, y+z$. Then $\mathcal{A}$ has five rank one elements and six rank two elements; of the rank two elements, four have $\mu = 1$, while two have $\mu = 2$. Thus,

$$\pi(\mathcal{A}, t) = 1 + 5t + 8t^2 + 4t^3.$$

Stanley [8] notes that there are two methods used for computing the Poincare polynomial; either by working directly from the definition above, or by working over a finite field, and counting (Stanley attributes the latter technique to Crapo/Rota/Orlik/Terao and Athanasiadis). One of the results of this paper is a new technique for computing $\pi(\mathcal{A}, t)$ for a three arrangement. First, we need a few more definitions. $\mathcal{A}$ is *essential* if rank $L_{\mathcal{A}} =$ dim $\mathbf{V}$. Henceforth, all arrangements will be *essential* and *central*, and $\mathbf{V}$ will be K^3, with $char(K) = 0$. Let $R = K[x, y, z]$, and suppose $\mathcal{A}$ consists of d distinct hyperplanes in $\mathbf{V}$. We denote the module of K derivations of R by $Der_K R$, and we define $Q = \prod_1^d l_i$, where l_i are the (homogeneous) linear forms which vanish on the hyperplanes of $\mathcal{A}$.

Definition 1.4 *The Terao Module $D(\mathcal{A})$ of $\mathcal{A}$ is the submodule of $Der_K R$ defined by*

$$D(\mathcal{A}) = \{\theta \in Der_K R \mid \theta(Q) \in Q \cdot R\}$$

Recall Euler's relation:

$$Q = \frac{1}{\text{degree } Q} \cdot (x \cdot \frac{\partial Q}{\partial x} + y \cdot \frac{\partial Q}{\partial y} + z \cdot \frac{\partial Q}{\partial z}).$$

From this it follows that $D(\mathcal{A})$ is nonzero. In fact, the Euler derivation generates a free summand of $D(\mathcal{A})$, so we may write $D(\mathcal{A}) = R(-1) \oplus D_0(\mathcal{A})$, where $D_0(\mathcal{A})$ is the kernel of the Jacobian matrix J_Q of Q (see, for example, Yuzvinsky [9]). Since the Jacobian is a homogeneous ideal, the kernel is a graded module; the grading is inherited from the inclusion of the kernel as a submodule of R^3 (which is why the Euler derivation corresponds to $R(-1)$).

Since $D_0(\mathcal{A})$ is a graded R module, we have a corresponding sheaf $\mathcal{D}_0(\mathcal{A})$ on $\mathbf{P}^2$, which is just the sheafification of $D_0(\mathcal{A})$. The process of passing to a sheaf is crucial in what follows, because a certain module of finite length arises. This module would present difficulties if we worked in the category of modules; by passing to the category of coherent sheaves, we may ignore it.

A useful tool when working with arrangements is the method of deletion and restriction. Let H be a hyperplane of $\mathcal{A}$, and let $\mathcal{A}'$ be the arrangement consisting of all the hyperplanes of $\mathcal{A}$ except H. $\mathcal{A}''$ is the arrangement obtained by restricting $\mathcal{A}$ to H. $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is called a *triple* of arrangements. Q' and Q'' will denote the defining polynomials for $\mathcal{A}'$ and $\mathcal{A}''$.

Example 1.5 If $\mathcal{A}$ is as in Example 1.3, and H is the hyperplane $x = 0$, then $Q = x \cdot y \cdot z \cdot (x + y + z) \cdot (y + z)$, $Q' = y \cdot z \cdot (x + y + z) \cdot (y + z)$, and $Q'' = y \cdot z \cdot (y + z)$.

Theorem 1.6 *(Brylawski [1]) If $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a triple of arrangements, then*

$$\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t \cdot \pi(\mathcal{A}'', t).$$

We may always assume that the first hyperplane of $\mathcal{A}$ is x; when we preform deletion/restriction, we will assume that x is the distinguished hyperplane. In other words, $Q = \prod_{i=1}^{d} l_i$ with $l_1 = x$, so $Q' = \prod_{i=2}^{d} l_i$ has no l_i divisible by x (i.e. $Q' \notin <x>$).

2 Preliminary Lemmas

In this section we collect the lemmas which will be used in §3, where the main theorem is proved. The proof is via induction on the number of hyperplanes of $\mathcal{A}$; it makes use of the deletion/restriction lemma, along with the definition of $D_0(\mathcal{A})$ as the kernel of the Jacobian. The following short exact sequences will relate the sheaves $\mathcal{D}_0(\mathcal{A})$, $\mathcal{D}_0(\mathcal{A}')$, and $\mathcal{D}_0(\mathcal{A}'')$.

$$\text{I. } 0 \longrightarrow < J_Q : x^2 > /J_{Q'} \longrightarrow R(-2)/J_{Q'} \longrightarrow R(-2)/ < J_Q : x^2 > \longrightarrow 0$$

$$\text{II. } 0 \longrightarrow R(-2)/ < J_Q : x^2 > \longrightarrow R(-1)/ < J_Q : x > \longrightarrow R(-1)/ < J_Q : x, x > \longrightarrow 0$$

$$\text{III. } 0 \longrightarrow R(-1)/ < J_Q : x > \longrightarrow R/J_Q \longrightarrow R/ < J_Q, x > \longrightarrow 0$$

The exactness of II and III is immediate; the exactness of I follows if $J_{Q'} \subseteq < J_Q : x^2 >$ (which we show in Lemma 2.3).

Lemma 2.1 $J_Q = < x \cdot \frac{\partial Q'}{\partial y}, x \cdot \frac{\partial Q'}{\partial z}, x \cdot \frac{\partial Q'}{\partial x} + Q' > = < x \cdot \frac{\partial Q'}{\partial y}, x \cdot \frac{\partial Q'}{\partial z}, \frac{\partial Q}{\partial x} >$.

Proof. Immediate from the definition of Q and Q'. Notice that this implies that $< J_Q, x > = < Q', x >$. $\square$

Lemma 2.2 $< J_Q : x > = < \frac{\partial Q'}{\partial y}, \frac{\partial Q'}{\partial z}, \frac{\partial Q}{\partial x} >$.

Proof. Recall (or see [2]) that $< J_Q : x > = \frac{J_Q \cap x}{x}$. We write $J_Q \cap x$ as $(< x \cdot \frac{\partial Q'}{\partial y}, x \cdot \frac{\partial Q'}{\partial z} > + < \frac{\partial Q}{\partial x} >) \cap x$. Since $< x \cdot \frac{\partial Q'}{\partial y}, x \cdot \frac{\partial Q'}{\partial z} > \subseteq x$, we may apply the modular law [2] to distribute, obtaining

$$J_Q \cap x =< x \cdot \frac{\partial Q'}{\partial y}, x \cdot \frac{\partial Q'}{\partial z} > + < x \cap \frac{\partial Q}{\partial x} > .$$

But $\frac{\partial Q}{\partial x} = x \cdot \frac{\partial Q'}{\partial x} + Q'$, so since x does not divide Q', $< x \cap \frac{\partial Q}{\partial x} >= x \cdot \frac{\partial Q}{\partial x}$. Thus,

$$J_Q \cap x =< x \cdot \frac{\partial Q'}{\partial y}, x \cdot \frac{\partial Q'}{\partial z}, x \cdot \frac{\partial Q}{\partial x} > .$$

With the initial observation that $< J_Q : x >= \frac{J_Q \cap x}{x}$, this concludes the proof. □

Lemma 2.3 $< J_Q : x > \subseteq J_{Q'} \subseteq < J_Q : x^2 >$.

Proof. By Lemma 2.2,

$$< J_Q : x >=< \frac{\partial Q'}{\partial y}, \frac{\partial Q'}{\partial z}, \frac{\partial Q}{\partial x} > \subseteq < \frac{\partial Q'}{\partial x}, \frac{\partial Q'}{\partial y}, \frac{\partial Q'}{\partial z} > .$$

This follows since by the Euler relation

$$\frac{\partial Q}{\partial x} = x \cdot \frac{\partial Q'}{\partial x} + Q' = x \cdot \frac{\partial Q'}{\partial x} + \frac{1}{\text{degree } Q'} \cdot (x \cdot \frac{\partial Q'}{\partial x} + y \cdot \frac{\partial Q'}{\partial y} + z \cdot \frac{\partial Q'}{\partial z}).$$

Lemma 2.1 shows that $x^2 \cdot \frac{\partial Q'}{\partial y}, x^2 \cdot \frac{\partial Q'}{\partial z} \subseteq J_Q$, so we need to show that $x^2 \cdot \frac{\partial Q'}{\partial x} \in J_Q$. But $x \cdot (x \cdot \frac{\partial Q'}{\partial x} + Q') = x \cdot \frac{\partial Q}{\partial x} \in J_Q$, so since $x \cdot Q' = Q \in J_Q$, $x^2 \cdot \frac{\partial Q'}{\partial x} \in J_Q$. □

Lemma 2.4 $< J_Q : x^2 >= J_{Q'} + \frac{1}{x} \cdot (\frac{\partial Q'}{\partial y} \cdot \frac{\frac{\partial Q'}{\partial z}|_{x=0}}{G} - \frac{\partial Q'}{\partial z} \cdot \frac{\frac{\partial Q'}{\partial y}|_{x=0}}{G})$, *where* $G = GCD(\frac{\partial Q'|_{x=0}}{\partial y}, \frac{\partial Q'|_{x=0}}{\partial z})$.

Proof. As in the proof of Lemma 2.2, we have

$$< J_Q : x^2 >= \frac{< J_Q : x > \cap x}{x} = \frac{< \frac{\partial Q'}{\partial y}, \frac{\partial Q'}{\partial z} > \cap x + x \frac{\partial Q'}{\partial x}}{x}.$$

So, we need to obtain $< \frac{\partial Q'}{\partial y}, \frac{\partial Q'}{\partial z} > \cap x$. It is clear that it contains $x \frac{\partial Q'}{\partial y}$ and $x \frac{\partial Q'}{\partial z}$. If we write

$$\begin{aligned}\frac{\partial Q'}{\partial y} &= p_1 x + p_2,\\ \frac{\partial Q'}{\partial z} &= q_1 x + q_2,\end{aligned}$$

where $p_2 = \frac{\partial Q'}{\partial y}|_{x=0}$ and $q_2 = \frac{\partial Q'}{\partial z}|_{x=0}$, then the remaining generator of $< \frac{\partial Q'}{\partial y}, \frac{\partial Q'}{\partial z} > \cap x$ is $\frac{p_2}{G} \cdot (q_1 x + q_2) - \frac{q_2}{G} \cdot (p_1 x + p_2)$. Note that $\frac{1}{x} \cdot (\frac{\partial Q'}{\partial y} \cdot \frac{\frac{\partial Q'}{\partial z}|_{x=0}}{G} - \frac{\partial Q'}{\partial z} \cdot \frac{\frac{\partial Q'}{\partial y}|_{x=0}}{G}) = \frac{q_1 \frac{\partial Q'}{\partial y} - p_1 \frac{\partial Q'}{\partial z}}{G}$. $\square$

Lemma 2.5 *Let p be the ideal of the point $(0:0:1) \in \mathbf{P}^2$, and write $Q = L_0 L_P$, where $L_P = \prod(a_i y + b_i x)$ and $L_0 = \prod(z + c_i y + d_i x)$. Then $(J_Q)_p =< \frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y} >_p$.*

Proof.

$$\begin{aligned}\frac{\partial Q}{\partial x} &= \frac{\partial L_P}{\partial x} L_0 + \frac{\partial L_0}{\partial x} L_P\\ \frac{\partial Q}{\partial y} &= \frac{\partial L_P}{\partial y} L_0 + \frac{\partial L_0}{\partial y} L_P\\ \frac{\partial Q}{\partial z} &= \frac{\partial L_P}{\partial z} L_0 + \frac{\partial L_0}{\partial z} L_P,\end{aligned}$$

Since $\frac{\partial L_p}{\partial z}$ vanishes, and both L_0 and $\frac{\partial L_0}{\partial z}$ are units in R_p, reducing and applying the Euler relation yields the result. $\square$

Lemma 2.6 *Let $L'_P = \prod_{i=1}^{m-1}(y + a_i x)$, $L_P = x \cdot L'_P$, a_i distinct. Then the sequence*

$$0 \longrightarrow R/ < \frac{\partial L'_P}{\partial x}, \frac{\partial L'_P}{\partial y} > \xrightarrow{\cdot x^2} R(2)/ < \frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y} > \longrightarrow R(2)/ < \frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y}, x^2 > \longrightarrow 0$$

is exact.

Proof. The sequence

$$0 \longrightarrow R/ < (\frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y}) : x^2 > \xrightarrow{\cdot x^2} R(2)/ < \frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y} > \longrightarrow R(2)/ < \frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y}, x^2 > \longrightarrow 0$$

is exact. Since L_P has no repeat factors, the partials are relatively prime (similarly for L'_P), so $< \frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y} >$ and $< \frac{\partial L'_P}{\partial x}, \frac{\partial L'_P}{\partial y} >$ are complete intersections, and the Hilbert series for the respective quotients (before twisting) are given by

$$\frac{1-2t^{m-1}+t^{2m-2}}{(1-t)^3} \quad \text{and} \quad \frac{1-2t^{m-2}+t^{2m-4}}{(1-t)^3}.$$

The relation

$$< \frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y}, x^2 >=< x^2, xy^{m-2}, y^{m-1} >$$

implies that the Hilbert series of $R/ < \frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y}, x^2 >$ is

$$\frac{1-t^2-2t^{m-1}+2t^m}{(1-t)^3},$$

so by additivity, the Hilbert series of $R/ < (\frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y}) : x^2 >$ is

$$\frac{t^{-2}-2t^{m-3}+t^{2m-4}}{(1-t)^3} - \frac{t^{-2}-1-2t^{m-3}+2t^{m-2}}{(1-t)^3} = \frac{1-2t^{m-2}+t^{2m-4}}{(1-t)^3}.$$

An easy check shows that $< \frac{\partial L'_P}{\partial x}, \frac{\partial L'_P}{\partial y} > \subseteq < (\frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y} >: x^2 >$, so we have

$$R/ < \frac{\partial L'_P}{\partial x}, \frac{\partial L'_P}{\partial y} > \twoheadrightarrow R/ < (\frac{\partial L_P}{\partial x}, \frac{\partial L_P}{\partial y}) : x^2 > .$$

Comparing the Hilbert series shows that this must be an isomorphism. □

Lemma 2.7 *$< J_Q : x^2 > /J_{Q'}$ is a module of finite length.*

Proof. We need to show that $(< J_Q : x^2 > /J_{Q'})_p = 0$ for any codimension two associated prime p of $J_{Q'}$. In fact, since both x and G annihilate $< J_Q : x^2 > /J_{Q'}$ (by Lemma 2.3

and the remarks at the end of Lemma 2.4), p must correspond to a singularity of Q' lying along x, so we may assume that $p = < x, y >$. Patching together the short exact sequence I (twisted by 2) with the short exact sequence

$$0 \longrightarrow R/ < J_Q : x^2 > \xrightarrow{\cdot x^2} R(2)/J_Q \longrightarrow R(2)/ < J_Q, x^2 > \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow < J_Q : x^2 > /J_{Q'} \xrightarrow{\phi} R/J_{Q'} \xrightarrow{\cdot x^2} R(2)/J_Q \longrightarrow R(2)/ < J_Q, x^2 > \longrightarrow 0.$$

Localize this sequence at p and apply Lemma 2.5. Comparing this sequence to the localization (at p) of the sequence of Lemma 2.6 shows that ϕ_p must be the zero map, hence $(< J_Q : x^2 > /J_{Q'})_p$ vanishes. $\square$

In order to apply Brylawski's theorem, we must obtain $D_0(\mathcal{A}')$ and $D_0(\mathcal{A}'')$. If we simply take $Q'|_{x=0}$, we do not obtain Q''; what we obtain is Q'' with some factors repeated (see Example 1.5). However, we have:

Lemma 2.8 $Q'' = \sqrt{Q'|_{x=0}}$

A two arrangement $\mathcal{B}$ is *completely* determined by the number of distinct hyperplanes (all two arrangements are free, and a two arrangement consisting of d distinct hyperplanes has $\pi(\mathcal{B}, t) = (1 + (d-1)t) \cdot (1 + t)$). For such a two arrangement, the Jacobian is a complete intersection of two polynomials of degree $d-1$, so $D_0(\mathcal{B})$ is isomorphic to $R(-d+1)$ (recall how the grading works), and $c_t(\mathcal{D}_0(\mathcal{B})^\vee) = 1 + (d-1)t$, as desired. Thus, to understand

$\mathcal{D}_0(\mathcal{A}'')$, we only need to obtain the degree of Q''. This is encoded by the short exact sequence II.

Lemma 2.9 *The resolution of* $R(-1)/<J_Q : x, x>$ *is given by:*

$$R(-2d+2+a) \stackrel{\phi_3}{\to} \begin{array}{c} R^2(-d) \\ \oplus \\ R(-2d+3+a) \end{array} \stackrel{\phi_2}{\to} \begin{array}{c} R^2(-d+1) \\ \oplus \\ R(-2) \end{array} \stackrel{\phi_1}{\to} R(-1) \to R(-1)/<J_Q : x, x> \to 0,$$

where $d-1-a$ *is the degree of* Q''.

Proof. The radical of a single polynomial may be obtained by dividing it by the greatest common divisor of the partials (see [2]), so if $G = GCD(\frac{\partial Q'|_{x=0}}{\partial y}, \frac{\partial Q'|_{x=0}}{\partial z})$, then

$$Q'' = \frac{Q'|_{x=0}}{G}.$$

This relates to the resolution in the following way. Let a denote the degree of G. By Lemma 2.2, $< J_Q : x >=< \frac{\partial Q'}{\partial y}, \frac{\partial Q'}{\partial z}, \frac{\partial Q}{\partial x} >$. Reducing by x allows us to kill off $\frac{\partial Q}{\partial x}$ (just expand it, and then use the Euler relation), so that

$$< J_Q : x, x >=< \frac{\partial Q'}{\partial y}|_{x=0}, \frac{\partial Q'}{\partial z}|_{x=0}, x > .$$

The structure of the resolution is now clear. The map ϕ_1 is $< J_Q : x, x >$; if we write:

$$\frac{\partial Q'}{\partial y}|_{x=0} = Gu$$

$$\frac{\partial Q'}{\partial z}|_{x=0} = Gv,$$

then

$$\phi_2 = \begin{matrix} v & x & 0 \\ -u & 0 & x \\ 0 & -Gu & -Gv \end{matrix}$$

and $\phi_3 = (x, -v, u)^t$. $\square$

Corollary 2.10 *$Ext^2(R(-1)/ < J_Q : x, x >, R) \simeq R(2d-2-a)/ < x, G >$, where G is as in Lemma 2.9.*

Proof.

$$\ker(\phi_3^t) = \begin{matrix} 0 & v & u \\ u & x & 0 \\ v & 0 & -x \end{matrix} \qquad \mathrm{im}(\phi_2^t) = \begin{matrix} v & -u & 0 \\ x & 0 & -Gu \\ 0 & x & -Gv \end{matrix}$$

$\square$

3 Main Theorem

Theorem 3.1 *If $\mathcal{A}$ is a three arrangement, then $\pi(\mathcal{A}, t) = c_t(\mathcal{D}_0(\mathcal{A})^\vee) \cdot (1+t)$.*

As noted in §2, the proof will be by induction on the number of hyperplanes of $\mathcal{A}$, and will use the deletion/restriction lemma. The short exact sequences of §2 relate the sheaves $\mathcal{D}_0(\mathcal{A})$, $\mathcal{D}_0(\mathcal{A}')$, and $\mathcal{D}_0(\mathcal{A}'')$. We first recall a few facts about the Chern polynomial; for more information, see Hartshorne ([4]).

Lemma 3.2 *If* $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$ *is a short exact sequence of sheaves, then*

$$c_t(\mathcal{B}) = c_t(\mathcal{A}) \cdot c_t(\mathcal{C}).$$

Lemma 3.3 $c_t(\mathcal{O}_{\mathbf{P}^n}(d)) = 1 + dt$.

Lemma 3.4 *If* $\mathcal{M}$ *is a sheaf on* $\mathbf{P}^n$, *then* $c_t(\mathcal{M}) \in \mathbf{Z}[t]/t^{n+1}$.

Lemma 3.5 *If* $\mathcal{F}$ *is a sheaf on* $\mathbf{P}^2$, *and if*

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^{m} \mathcal{O}(a_i) \longrightarrow \bigoplus_{j=1}^{m-2} \mathcal{O}(b_j) \longrightarrow 0,$$

then $c_t(\mathcal{F}(1)) = c_t(\mathcal{F}) + 2t + (\sum a_i - \sum b_j + 1)t^2$.

Proof. (Main Theorem) We exploit two facts about the short exact sequences of §2. First, none of the modules appearing in those exact sequences has support in codimension one. Thus, when we take the long exact sequences of $Ext^*(-, R)$ modules, the Ext^1's all vanish. Second, once we sheafify the sequences of Ext modules, the Ext^3's all vanish (since they have support in codimension three, see [3]). To prove the theorem, it suffices to show

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = c_t(\mathcal{D}_0(\mathcal{A}')^\vee) + t c_t(\mathcal{D}_0(\mathcal{A}'')^\vee).$$

Let Q, Q', and Q'' be the respective defining polynomials, and let G be as in Lemma 2.9. The cokernel of the Jacobian of $\mathcal{A}$ has a resolution of the form:

$$0 \longrightarrow \bigoplus_{j=1}^{m-2} R(-\beta_j) \xrightarrow{\phi_3} \bigoplus_{i=1}^{m} R(-\alpha_i) \xrightarrow{\phi_2} R^3 \xrightarrow{\phi_1} R(d-1) \longrightarrow R(d-1)/J_Q \longrightarrow 0.$$

Dualizing, computing homology, and sheafifying (henceforth, *all* modules will be sheafified), we obtain the short exact sequence

$$0 \longrightarrow im(\phi_2^t) \longrightarrow \mathcal{D}_0(\mathcal{A})^\vee \longrightarrow Ext^2(R(d-1)/J_Q, R) \longrightarrow 0.$$

This yields

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = c_t(im(\phi_2^t))c_t(Ext^2(R(d-1)/J_Q, R)).$$

Since $Ext^1(R(d-1)/J_Q, R)$ vanishes,

$$im(\phi_2^t) \simeq (R^3)^\vee / ker(\phi_2^t) \simeq R^3/im(\phi_1^t) \simeq R^3/J_Q^t,$$

and thus $c_t(im(\phi_2^t)) = \frac{1}{1+(1-d)t}$. Next, we need to consider $c_t(Ext^2(R(d-1)/J_Q, R))$. From the short exact sequence III and the vanishing of Ext^1 and Ext^3, we have

$$0 \longrightarrow Ext^2(R(d-1)/<x, Q'>, R) \longrightarrow Ext^2(R(d-1)/J_Q, R) \longrightarrow Ext^2(R(d-2)/<J_Q : x>, R) \longrightarrow 0.$$

Applying Lemma 3.2 to this, and consolidating the expression, we find that

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = \frac{1}{1+(1-d)t} \cdot c_t(Ext^2(R(d-1)/<x, Q'>, R)) \cdot c_t(Ext^2(R(d-2)/<J_Q : x>, R)).$$

Lemma 3.6 $c_t(Ext^2(R(d-1)/<x, Q'>, R)) = \frac{(1+t)(1+(1-d)t)}{1+(2-d)t}$.

Proof. x does not divide Q', so $<x, Q'>$ is a complete intersection, with resolution

$$0 \longrightarrow R(-1) \longrightarrow R(d-2) \oplus R \longrightarrow R(d-1) \longrightarrow R(d-1)/<x, Q'> \longrightarrow 0.$$

Since $Ext^2(R(d-1)/<x, Q'>, R) \simeq R(1)/<x, Q'>$, to compute $c_t(Ext^2(R(d-1)/<x, Q'>, R))$, we tensor the above exact sequence by $R(-d+2)$ and apply Lemma 3.2. $\square$

The short exact sequence II, the vanishing of the Ext^1 and Ext^3 modules, and Lemma 3.2 shows that $c_t(Ext^2(R(d-2)/ < J_Q : x >, R))$ is equal to:

$$c_t(Ext^2(R(d-2)/ < J_Q : x, x >, R)) \cdot c_t(Ext^2(R(d-3)/ < J_Q : x^2 >, R)).$$

Lemma 3.7 $c_t(Ext^2(R(d-2)/ < J_Q : x, x >, R)) = 1 - at^2$, *where* a *is the degree of* G *in Lemma 2.9.*

Proof. By Corollary 2.10, $Ext^2(R(d-2)/ < J_Q : x, x >, R) \simeq R(d-1-a)/ < x, G >$. Since $< x, G >$ is a complete intersection, take the obvious resolution (twisted by $d-1-a$) and compute. (If $G = 1$, the module vanishes, and the formula for c_t yields 1) $\square$

Putting together everything, we have:

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = \frac{1}{1+(1-d)t} \cdot \frac{(1+t)(1+(1-d)t)}{1+(2-d)t} \cdot (1-at^2) \cdot c_t(Ext^2(R(d-3)/ < J_Q : x^2 >, R)).$$

Lemma 3.8 $c_t(Ext^2(R(d-3)/ < J_Q : x^2 >, R)) = \frac{1+(d-1)t}{(1+t)^3} \cdot c_t(\mathcal{D}_0(\mathcal{A}')^\vee(1))$.

Proof. Lemma 2.7 and the short exact sequence I imply that

$$c_t(Ext^2(R(d-3)/ < J_Q : x^2 >, R)) = c_t(Ext^2(R(d-3)/J_{Q'}, R)).$$

Proceeding exactly as we did to obtain the identity

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = c_t(im(\phi_2^t))c_t(Ext^2(R(d-1)/J_Q, R)),$$

we obtain (but with a degree shift of one):

$$c_t(\mathcal{D}_0(\mathcal{A}')^\vee(1)) = \frac{(1+t)^3}{1+(d-1)t} c_t(Ext^2(R(d-3)/J_{Q'}, R)),$$

hence

$$c_t(Ext^2(R(d-3)/J_{Q'},R)) = \frac{1+(d-1)t}{(1+t)^3} \cdot c_t(\mathcal{D}_0(\mathcal{A}')^\vee(1)).$$

□

Combining Lemma 3.8 and the formula directly above it, and simplifying, we obtain

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = \frac{(1+(d-1)t)(1-at^2)}{(1+t)^2(1+(2-d)t)} \cdot c_t(\mathcal{D}_0(\mathcal{A}')^\vee(1));$$

expanding the fractional part, and reducing modulo t^3 yields

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = (1-t+(d-a-1)t^2) \cdot c_t(\mathcal{D}_0(\mathcal{A}')^\vee(1)).$$

By Lemma 3.5, if

$$0 \longrightarrow \mathcal{D}_0(\mathcal{A}')^\vee \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(\hat{a}_i) \longrightarrow \bigoplus_{j=1}^{n-2} \mathcal{O}(\hat{b}_j) \longrightarrow 0,$$

then

$$c_t(\mathcal{D}_0(\mathcal{A}')^\vee(1)) = c_t(\mathcal{D}_0(\mathcal{A}')^\vee) + 2t + (\sum \hat{a}_i - \sum \hat{b}_j + 1)t^2.$$

Thus,

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = (1-t+(d-a-1)t^2) \cdot (c_t(\mathcal{D}_0(\mathcal{A}')^\vee) + 2t + (\sum \hat{a}_i - \sum \hat{b}_j + 1)t^2).$$

Multiplying out, we have

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = c_t(\mathcal{D}_0(\mathcal{A}')^\vee) + (2-c_t(\mathcal{D}_0(\mathcal{A}')^\vee))t + (c_t(\mathcal{D}_0(\mathcal{A}')^\vee)(d-a-1) + \sum \hat{a}_i - \sum \hat{b}_j - 1)t^2.$$

If we use the relation that

$$c_t(\mathcal{D}_0(\mathcal{A}')^\vee) = 1 + (\sum \hat{a}_i - \sum \hat{b}_j)t + (\sum_{r<s} \hat{a}_r \hat{a}_s + \sum_{t<u} \hat{b}_t \hat{b}_u + \sum \hat{b}_j^{\,2} - \sum_{s,u} \hat{a}_s \hat{b}_u)t^2,$$

then when we reduce modulo t^3, we obtain

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = c_t(\mathcal{D}_0(\mathcal{A}')^\vee) + t + (d - a - 2)t^2.$$

As noted in §2, since degree $Q'' = d - 1 - a$, $c_t(\mathcal{D}_0(\mathcal{A}'')^\vee) = 1 + (d - a - 2)t$; so the above expression simplifies to:

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = c_t(\mathcal{D}_0(\mathcal{A}')^\vee) + t c_t(\mathcal{D}_0(\mathcal{A}'')^\vee).$$

By induction on the number of hyperplanes in $\mathcal{A}$, this proves the theorem. □

4 Algorithm and Examples

Theorem 4.1 *If $\mathcal{A}$ is a three arrangement, and the resolution for the cokernel of the Jacobian is given by*

$$0 \longrightarrow \bigoplus_{j=1}^{m-2} R(-\beta_j) \xrightarrow{\phi_3} \bigoplus_{i=1}^{m} R(-\alpha_i) \xrightarrow{\phi_2} R^3 \xrightarrow{\phi_1} R(d-1) \longrightarrow R(d-1)/J_Q \longrightarrow 0, \text{ then}$$

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = 1 + (\sum_{i=1}^{m} \alpha_i - \sum_{j=1}^{m-2} \beta_j)t + (\sum_{1\le r<s\le m} \alpha_r\alpha_s + \sum_{1\le t\le u\le m-2} \beta_t\beta_u - \sum_{i=1,j=1}^{m,m-2} \alpha_i\beta_j)t^2.$$

Proof. Truncating the above exact sequence, we obtain:

$$0 \longrightarrow \bigoplus_{j=1}^{m-2} R(-\beta_j) \xrightarrow{\phi_3} \bigoplus_{i=1}^{m} R(-\alpha_i) \xrightarrow{\phi_2} D_0(\mathcal{A}) \longrightarrow 0.$$

From this short exact sequence, we obtain a long exact sequence of Ext modules:

$$0 \longrightarrow D_0(\mathcal{A})^\vee \longrightarrow \bigoplus_{i=1}^{m} R(\alpha_i) \longrightarrow \bigoplus_{j=1}^{m-2} R(\beta_j) \longrightarrow Ext^1(D_0(\mathcal{A}), R) \longrightarrow 0.$$

Tracing through the exact sequences, we find that $Ext^1(D_0(\mathcal{A}), R) \simeq Ext^3(R(d-1)/J_Q, R)$. As noted in §3, $Ext^3(R(d-1)/J_Q, R)$ is supported only at $< x, y, z >$, and so when we sheafify the sequence, it disappears, yielding the short exact sequence:

$$0 \longrightarrow \mathcal{D}_0(\mathcal{A})^\vee \longrightarrow \bigoplus_{i=1}^{m} \mathcal{O}(\alpha_i) \longrightarrow \bigoplus_{j=1}^{m-2} \mathcal{O}(\beta_j) \longrightarrow 0.$$

Since the map from $\bigoplus\limits_{i=1}^{m} \mathcal{O}(\alpha_i)$ to $\bigoplus\limits_{j=1}^{m-2} \mathcal{O}(\beta_j)$ which is induced by ϕ_3^t has no cokernel, we can split the map, so $\mathcal{D}_0(\mathcal{A})^\vee$ is a vector bundle on $\mathbf{P}^2$. Using the properties of the Chern polynomial introduced in the last section, we obtain

$$c_t(\mathcal{D}_0(\mathcal{A})^\vee) = \frac{c_t(\bigoplus\limits_{i=1}^{m} \mathcal{O}(\alpha_i))}{c_t(\bigoplus\limits_{j=1}^{m-2} \mathcal{O}(\beta_j))} = \frac{\prod\limits_{i=1}^{m}(1+\alpha_i t)}{\prod\limits_{j=1}^{m-2}(1+\beta_j t)} = \prod_{i=1}^{m}(1+\alpha_i t)\prod_{j=1}^{m-2}(1-\beta_j t+\beta_j^2 t^2) \bmod t^3.$$

□

Corollary 4.2 *If $\mathcal{A}$ is a three arrangement, with Jacobian as above, then*

$$\pi(\mathcal{A}, t) = (1+t)\cdot(1+(\sum_{i=1}^{m}\alpha_i - \sum_{j=1}^{m-2}\beta_j)t + (\sum_{1\le r<s\le m}\alpha_r\alpha_s + \sum_{1\le t\le u\le m-2}\beta_t\beta_u - \sum_{i=1,j=1}^{m,m-2}\alpha_i\beta_j)t^2).$$

Example 4.3 For the arrangement of Example 1.3, the resolution is:

$$0 \longrightarrow R^2(-2) \longrightarrow R^3 \longrightarrow R(4) \longrightarrow R(4)/J_Q \longrightarrow 0.$$

Thus, we see that the Chern polynomial of $\mathcal{D}_0(\mathcal{A})^\vee$ is just $1+4t+4t^2 = (1+2t)^2$.

Example 4.4 Consider the arrangement with defining polynomial $Q = xyz(x+y)(y+z)(z+x)(x+2y)(y+2z)(z+2x)$. The resolution of the Jacobian of Q is:

$$0 \longrightarrow R(-7) \longrightarrow R^3(-5) \longrightarrow R^3 \longrightarrow R(8) \longrightarrow R(8)/J_Q \longrightarrow 0.$$

We obtain $c_t(\mathcal{D}_0(\mathcal{A})^\vee) = 1 + 8t + 19t^2$.

Concluding remarks This algorithm is implemented as a Macaulay2 script called *hypertool*, available via anonymous ftp at *cam.cornell.edu*, in the directory */pub/schenck*. Macaulay2 itself is available at *http://www.math.uiuc.edu/Macaulay2*. We are in the process of extending these results to the $\mathbf{P}^n$ case; the situation becomes quite a bit more complicated for several reasons. First, in general $\mathcal{D}_0(\mathcal{A})^\vee$ is not a vector bundle; this means that in the formula for $c_t(\mathcal{D}_0(\mathcal{A})^\vee)$, the Chern polynomials of certain Ext modules will arise. Also, the higher dimensional analogs of the lemmas used here become much more complicated. Nevertheless, there seems to be quite a bit that can be proved in the case where n is arbitrary.

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