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Cohomology vanishing and a problem in approximation theory

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Abstract. For a simplicial subdivison Δ of a region in k^n (k algebraically closed) and $r \in \mathbf{N}$, there is a reflexive sheaf $\mathcal{K}$ on $\mathbf{P}^n$, such that $H^0(\mathcal{K}(d))$ is essentially the space of piecewise polynomial functions on Δ, of degree at most d, which meet with order of smoothness r along common faces. In [9], Elencwajg and Forster give bounds for the vanishing of the higher cohomology of a bundle $\mathcal{E}$ on $\mathbf{P}^n$ in terms of the top two Chern classes and the generic splitting type of $\mathcal{E}$. We use a spectral sequence argument similar to that of [16] to characterize those Δ for which $\mathcal{K}$ is actually a bundle (which is always the case for $n = 2$). In this situation we can obtain a formula for $H^0(\mathcal{K}(d))$ which involves only local data; the results of [9] cited earlier allow us to give a bound on the d where the formula applies. We also show that a major open problem in approximation theory may be formulated in terms of a cohomology vanishing on $\mathbf{P}^2$ and we discuss a possible connection between semi-stability and the conjectured answer to this open problem.

Introduction

The space of r-differentiable piecewise polynomial functions (splines) on a simplicial subdivision Δ of some region in k^n is an object of fundamental interest in approximation theory; a nice introduction to the use of algebro-geometric methods in this area may be found in Cox, Little and O'Shea [6]. Since the smoothness condition is local, it is natural to define the question in terms of a sheaf $\mathcal{K}$, an approach taken by Stiller in [21] and Lau and Stiller [13]. $\mathcal{K}$ is reflexive, but not (in general) a bundle when $n > 2$. We relate this approach to Billera's use of simplicial homology [4], and use a spectral sequence argument similar to that of [16] to characterize those Δ for which $\mathcal{K}$ is a bundle, obtaining a formula for $H^0(\mathcal{K}(d))$. In fact, we obtain the following general result:

Theorem. *Let*

$$\mathcal{C} : 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

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be a complex of graded modules on $k[x_0, \ldots, x_n]$*, with* C_i *Cohen–Macaulay, supported in codimension* $n - i$*. If*

$$H_i(\mathcal{C}) \text{ is supported in codimension } \geq n + 2 - i, \forall i < n$$

then (the sheaf associated to) $H_n(\mathcal{C})$ *is a bundle iff* $H_i(\mathcal{C})$ *is of finite length for all* $i < n$*.*

By restricting a long exact sequence in which $\mathcal{K}$ appears to a line, we are able to apply results of Shatz to obtain constraints on the splitting type of $\mathcal{K}$. In [9] Elencwagj and Forster give bounds on the vanishing of the cohomology of a bundle; in the proof they apply a result of Schneider which requires $k = \mathbf{C}$. Coandă [5] gives an alternate proof, which only requires k algebraically closed, which we will assume throughout the paper. Combining the results which describe when $\mathcal{K}$ is a bundle with the work of Elencwajg and Forster, we obtain a bound showing where our formula for $H^0(\mathcal{K}(d))$ applies.

On $\mathbf{P}^2$ (where reflexive sheaves are always bundles), the bound we obtain is sometimes not as good as the bound obtained by Alfeld and Schumaker in [2]. One situation where we can sometimes better the Alfeld–Schumaker bound is when we know the generic splitting type; we study such situations and the connection to semi-stability in some detail in Sect. 2. In fact, our methods seem to point to a connection between (semi-)stability and a conjecture concerning the dimension of the space of splines of degree at most d for $d \geq 2r + 1$ in dimension $n = 2$. Alfeld [1] has noted that the $r = 1$ case of this conjecture is probably the most famous outstanding question in bivariate splines.

For $n > 2$, not much is known about the dimension of the spline space. In [16] Schenck shows that the top three coefficients of the Hilbert polynomial of $\mathcal{K}$ can be determined (for any Δ) from the combinatorics and the local geometry; in [3], Alfeld, Schumaker and Whiteley analyze the $n = 3$, $r = 1$ case for generic Δ. The point is that for $n > 2$, our results and bounds apply to previously unknown cases; in particular, algebro-geometric results have something useful to say about approximation theory.

This paper is structured as follows. In the first section, we give definitions and review previous results. In Sect. 2, we warm up by applying these results to the $\mathbf{P}^2$ case. In Sect. 3, we give the spectral sequence argument which shows when $\mathcal{K}$ is a bundle. In the final section, we apply the methods to the higher dimensional case. We thank the Mathematical Sciences Research Institute in Berkeley, California, where our collaboration began.

1. Preliminaries

We begin with the results we need from algebraic geometry. In [20], Shatz proves the following criterion for sub-bundles of bundles on $\mathbf{P}^1$: for a sequence of integers $a_1 \geq a_2 \geq \ldots \geq a_n$, define the Harder–Narasimhan polygon of $a_1 \geq a_2 \geq \ldots \geq a_n$ as the convex polygon

$$\mathrm{HNP}(a_1 \geq a_2 \geq \ldots \geq a_n) = \mathrm{conv}\left\{(0,0), (1, a_1), (2, a_1 + a_2), \ldots \left(n, \sum_{i \leq n} a_i\right)\right\}.$$

Theorem 1.1 (Shatz, [20]**).** *Let* $V = \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^1}(a_i)$ *and* $W = \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^1}(b_i)$. *Suppose* $\sum a_i = \sum b_i$ *and* $\mathrm{HNP}(b_1 \geq b_2 \geq \ldots \geq b_m) \geq \mathrm{HNP}(a_1 \geq a_2 \geq \ldots \geq a_m)$. *If* $M = \{l \mid b_l \leq a_l\}$, *then there is an exact sequence*

$$0 \longrightarrow \bigoplus_{m \in M} \mathcal{O}_{\mathbf{P}^1}(b_m) \longrightarrow V \longrightarrow \bigoplus_{m \notin M} \mathcal{O}_{\mathbf{P}^1}(b_m) \longrightarrow 0.$$

Next, we review the work of Elencwajg and Forster. Let $\mathcal{E}$ be a rank r bundle on $\mathbf{P}^n$, and suppose for a generic line L that $\mathcal{E}|_L \cong \bigoplus_{i=1}^{r} \mathcal{O}_L(a_i)$, with $a_i \geq a_{i+1}$. Put $c_2^L(\mathcal{E}) = c_2(\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{P}^n}(a_i))$. Define $\delta(\mathcal{E}) = c_2(\mathcal{E}) - c_2^L(\mathcal{E})$, $b(\mathcal{E}) = a_r$, and $e(\mathcal{E}) = a_1 - a_r$.

Theorem 1.2 (Elencwajg and Forster, [9] 2.18**).** *If* $\mathcal{E}$ *is a bundle on* $\mathbf{P}^2$, *then* $H^1(\mathcal{E}(d)) = 0$ *for all* $d \geq \delta(\mathcal{E}) - b(\mathcal{E}) - 1$ *and* $\mathcal{E}$ *is generated by global sections for* $d \geq \delta(\mathcal{E}) - b(\mathcal{E})$.

Theorem 1.3 (Elencwajg and Forster, [9] 3.3**).** *If* $\mathcal{E}$ *is a bundle on* $\mathbf{P}^n$, *then there exists a polynomial* $Q(c_1(\mathcal{E}), c_2(\mathcal{E}), e(\mathcal{E}))$ *such that* $H^q(\mathcal{E}(d)) = 0$ *for all* $d \geq Q(c_1(\mathcal{E}), c_2(\mathcal{E}), e(\mathcal{E}))$ *and all* $q \geq 1$.

Descriptions of all the objects defined in the remainder of this section may be found in [16] or [21], so we keep the presentation here brief. In [4], Billera introduced the use of simiplicial homology in order to solve a conjecture of Strang. Let Δ be a pure, strongly connected, topologically trivial n-dimensional simplicial complex, embedded in k^n (think of Δ as triangulating the interior of a n-polytope), and let Δ_i^0 be the set of interior i-faces of Δ (all n dimensional faces are considered interior). $C_d^r(\Delta)$ will denote the space of C^r piecewise polynomial functions on Δ of degree at most d (think of assigning a polynomial of degree at most d to each maximal simplex of Δ in such a way that polynomials on adjacent simplices meet C^r smoothly).

Let $R = k[x_0 \ldots x_n]$. Re-embed Δ in the plane $\{x_n = 1\} \subseteq k^{n+1}$, and form the cone $\hat{\Delta}$ of Δ with the origin. The following construction encodes all the spaces $C_d^r(\Delta)$ into a single graded R module $C^r(\hat{\Delta})$ (i.e. $C^r(\hat{\Delta})_d \simeq C_d^r(\Delta)$).

For τ an $n-1$ face of Δ, let l_τ be a nonzero linear form vanishing on $\hat{\tau}$, and for an i-face ξ define

$$J(\xi) = \sum_{\xi \in \tau} < l_\tau^{r+1} > .$$

As in [16], define a chain complex of graded modules

$$\mathcal{R}/\mathcal{J} : \cdots \longrightarrow \bigoplus_{\alpha \in \Delta_{i+1}^0} R/J(\alpha) \xrightarrow{\partial_{i+1}} \bigoplus_{\beta \in \Delta_i^0} R/J(\beta) \xrightarrow{\partial_i} \bigoplus_{\gamma \in \Delta_{i-1}^0} R/J(\gamma) \xrightarrow{\partial_{i-1}} \cdots,$$

where ∂_i is the usual boundary operator in relative (modulo $\partial\Delta$) homology. If we write $\mathcal{R}/\mathcal{J}$ as the quotient of the short exact sequence of complexes

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}/\mathcal{J} \longrightarrow 0,$$

then $C^r(\hat{\Delta}) \cong H_n(\mathcal{R}/\mathcal{J}) \cong R \bigoplus H_{n-1}(\mathcal{J})$; let $\mathcal{K}$ be the (reflexive) sheaf associated to the module $H_{n-1}(\mathcal{J})$. $\mathcal{K}$ is a bundle iff the sheaf associated to $C^r(\hat{\Delta})$ is a bundle; we will pass frequently between the equivalent categories of finitely generated graded modules and coherent sheaves.

2. The $\mathbf{P}^2$ case

In this section, we consider the planar case. From the remarks in the last section, the dimension of the space splines of degree d (for fixed r and Δ) is given by

$$\dim C^r_d(\Delta) = \dim H^0(\mathcal{K}(d)) + \binom{d+2}{2},$$

where $\binom{d+2}{2}$ comes from the dimension of the space of polynomials of degree at most d in two variables and $\mathcal{K}$ is a vector bundle of rank $f_2 - 1$, where f_2 is the number of triangles in Δ. In [2], Alfeld and Schumaker show that the dimension of $C^r_d(\Delta)$ on a triangulated planar domain Δ is given by a certain formula for $d \geq 3r + 1$; it is conjectured [15] that this formula also holds for $d \geq 2r + 1$, moreover this must be tight, as examples show that the formula fails for $d = 2r$. We call this the "$2r + 1$" conjecture. From [13], it follows that $\mathcal{K}$ can also be realized as the kernel of a map

$$\mathcal{O}_{\mathbf{P}^2}^{f_1^0}(-r-1) \to \mathcal{O}_{\mathbf{P}^2}^{f_0^0},$$

where f_1^0 and f_0^0 are the number of interior edges (resp. interior vertices) of Δ. An easy way to think of this is that a spline on the star of a vertex v is just a syzygy on $J(v)$. Intersecting the conditions yields an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbf{P}^2}^{f_1^0}(-r-1) \to \mathcal{O}_{\mathbf{P}^2}^{f_0^0} \to \mathcal{C} \to 0,$$

where $\mathcal{C}$ is a skyscraper sheaf supported on Δ_0^0; in fact $\mathcal{C}$ is just the sheaf associated to

$$\bigoplus_{v \in \Delta_0^0} R/J(v).$$

Taking Euler characteristics yields

$$\chi(\mathcal{K}(d)) = \chi\left(\mathcal{O}_{\mathbf{P}^2}^{f_1^0}(d-r-1)\right) + \chi(\mathcal{C}) - \chi\left(\mathcal{O}_{\mathbf{P}^2}^{f_0^0}(d)\right).$$

The right-hand side of this formula (plus $\binom{d+2}{2}$ for the constant splines) is the conjectured dimension for the space of splines of degree d and smoothness r on Δ, at least for $d \geq 2r+1$. The conjecture is thus equivalent to $H^1(\mathcal{K}(d)) = 0$ and $H^2(\mathcal{K}(d)) = 0$ for $d \geq 2r + 1$.

Lau and Stiller [13] have shown that $H^2(\mathcal{K}(d)) = 0$ for $d \geq 2r$; this also follows from the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigoplus_{v \in \Delta_0^0} \mathcal{K}_v \longrightarrow \mathcal{O}_{\mathbf{P}^2}^{f_1^{00}}(-r-1) \longrightarrow 0,$$

which appears later in this section. So we can interpret the “$2r+1$” conjecture for planar splines as a regularity condition on $\mathcal{K}$ ($\mathcal{K}$ is $2r+2$ regular) or simply as a cohomology vanishing condition:

Conjecture 2.1. The “$2r+1$” conjecture:

$$H^1(\mathcal{K}(d)) = 0 \quad \text{for} \quad d \geq 2r+1.$$

Using Bernstein–Bezier techniques, Alfeld and Schumaker [2] prove that the conjectured formula holds for $d \geq 3r+1$, which implies the cohomology vanishing for $d \geq 3r+1$.

$$H^1(\mathcal{K}(d)) = 0 \quad \text{for} \quad d \geq 3r+1.$$

Our strategy is to apply standard cohomology vanishing theorems (e.g. Elencwajg and Forster [9] and Hartshorne [12]) to the problem. In some instances, we can improve on the $3r+1$ bound of Alfeld and Schumaker. To do this, we need to know something about the generic splitting type of $\mathcal{K}$. Even without this, we can get a worst case estimate for the splitting type (using results in Shatz) and then can apply the vanishing theorems.

In order to state our results for the planar case, we need a bit more notation. For $v \in \Delta_0^0$, let ϵ_v denote the number of edges incident to v, k_v the number of those edges of distinct slope, and put $\alpha_v = \lfloor \frac{r+1}{k_v-1} \rfloor$. Let $\mathcal{K}_v$ denote the module of splines on star(v). In [19], Schumaker gave a dimension formula for the star, from which it follows that

$$\mathcal{K}_v \simeq \mathcal{O}_{\mathbf{P}^2}^{s_1}(-r-1-\alpha_v) \bigoplus \mathcal{O}_{\mathbf{P}^2}^{s_2}(-r-2-\alpha_v) \bigoplus \mathcal{O}_{\mathbf{P}^2}^{s_3}(-r-1),$$

where $s_1 = (k_v-1)\alpha_v + k_v - r - 2$, $s_2 = r+1-(k_v-1)\alpha_v$, and $s_3 = \epsilon_v - k_v$. Let f_1^{00} denote the number of totally interior edges (i.e. neither vertex lies on $\partial\Delta$), and $f_1^{0\partial}$ be the number of edges with one interior vertex and one boundary vertex. From Lemma 3.8 of [17] and the snake lemma, we obtain an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigoplus_{v\in\Delta_0^0} \mathcal{K}_v \longrightarrow \mathcal{O}_{\mathbf{P}^2}^{f_1^{00}}(-r-1) \longrightarrow 0. \qquad (*)$$

Since

$$\sum_{v\in\Delta_0^0} (\epsilon_v - 1) = 2f_1^{00} + f_1^{0\partial} - f_0^0,$$

it is obvious that rank $\mathcal{K} = f_1^0 - f_0^0 = f_2 - 1$, and a computation shows that

$$\begin{aligned} c_1(\mathcal{K}) &= -f_1^0(r+1), \\ c_2(\mathcal{K}) &= \binom{f_1^0}{2}(r+1)^2 - \binom{r+2}{2} f_0^0 \\ &\quad + \frac{1}{2}\sum_{v\in\Delta_0^0} \big((k_v-1)\alpha_v^2 + (k_v-2r-3)\alpha_v\big). \end{aligned}$$

From the exact sequence $(*)$ we see that if $\mathcal{K}|_L \simeq \bigoplus_{i=1}^{f_2-1} \mathcal{O}_L(a_i)$, then $a_i \leq -r-1$. Since the rank of $\mathcal{K}$ is $f_2 - 1$ it follows that

$$e(\mathcal{K}) \leq |c_1(\mathcal{K}) + (f_2 - 1)(r+1)| = f_0^0(r+1);$$

this same estimate also yields

$$b(\mathcal{K}) \geq -(f_0^0 + 1)(r+1).$$

From ([9], Sect. 2.6) we have

$$\delta(\mathcal{K}) \leq c_2(\mathcal{K}) - \frac{\operatorname{rank}(\mathcal{K}) - 1}{2 \cdot \operatorname{rank}(\mathcal{K})} c_1(\mathcal{K})^2 + \frac{\operatorname{rank}(\mathcal{K})}{8} e(\mathcal{K})^2,$$

so from the estimates above we obtain

Theorem 2.2. $H^1(\mathcal{K}(d)) = 0$ *if*

$$d \geq c_2(\mathcal{K}) - \frac{f_2 - 2}{2(f_2 - 1)}(f_1^0(r+1))^2 + \frac{f_2 - 1}{8}(f_0^0(r+1))^2 + (f_0^0 + 1)(r+1) - 1.$$

We can improve this bound by using the Shatz [20] constraints and the α_v, at the price of more complexity in the formula. We consider a particularly nice case. Call Δ uniform if $\epsilon_v = k_v$ is the same for all $v \in \Delta_0^0$, and divisible if $\frac{r+1}{k_v - 1} \in \mathbf{Z}$. Note that then $\alpha_v = \frac{r+1}{k_v-1}$. For such Δ, the bound has the form:

Theorem 2.3. *For a uniform, divisible* Δ, $H^1(\mathcal{K}(d)) = 0$ *if*

$$d \geq \alpha_v^2 \cdot \binom{f_1^{00} + 1}{2} + \alpha_v(k_v + f_1^{00}) - 1.$$

Proof. If Δ is uniform and divisible, then (in the above notation) $s_3 = s_2 = 0$ and $s_1 = k_v - 1$, so we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathbf{P}^2}^{f_0^0(k_v-1)}(-r-1-\alpha_v) \longrightarrow \mathcal{O}_{\mathbf{P}^2}^{f_1^{00}}(-r-1) \longrightarrow 0,$$

Thus,

$$c_2(\mathcal{K}) = \binom{f_1^0}{2}(r+1)^2 - f_0^0(r+1)\left(\frac{\alpha_v + r + 1}{2}\right).$$

The worst (in terms of vanishing) possible splitting type allowed by Shatz is then

$$\mathcal{K}|_L \simeq \mathcal{O}_L^{f_1^0 - f_d^0 - 1}(-r-1-\alpha_v) \bigoplus \mathcal{O}_L(\alpha_v(f_2 - 2) - (f_0^0 + 1)(r+1)).$$

Computing, we find that

$$\delta(\mathcal{K}) \leq \alpha_v^2 \cdot \left(\frac{f_0^0}{2} \cdot k_v \cdot (1 - 2f_1^0 + f_0^0 \cdot k_v) + \binom{f_1^0}{2}\right) = \alpha_v^2 \cdot \binom{f_1^{00} + 1}{2},$$

and

$$b(\mathcal{K}) = \alpha_v(f_2 - 2) - (f_0^0 + 1)(r+1) = -\alpha_v(k_v + f_1^{00}).$$

The theorem is then immediate from the Elencwajg–Forster result. □

To improve our bound, we note that the size of $H^1(\mathcal{K}(d))$ for $d \geq 2r+1$ depends only on the size of $H^1(\mathcal{E}(d))$ for a certain two bundle $\mathcal{E}$ constructed from $\mathcal{K}$. The additional vanishing for $2r+1 \leq d < 3r+1$ seems to be related to the semi-stability of $\mathcal{E}$. After splitting off line bundle summands from $\mathcal{K}$ (which do not contribute to H^1)

$$\mathcal{K} = \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbf{P}^2}(a_i) \bigoplus \mathcal{K}_1,$$

where $a_1 \geq \ldots \geq a_l$, we get a bundle $\mathcal{K}_1$ whose generic splitting type (at least when $\mathcal{K}_1$ is indecomposable) seems to be as balanced as possible. Namely

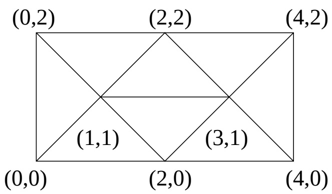
$$\mathcal{K}_1|_L = \bigoplus_{j=1}^{m} \mathcal{O}_L(b_j)$$

with $b_1 \geq \ldots \geq b_m$ and $b_1 - b_m \leq 1$ If we twist $\mathcal{K}_1$ by $\mathcal{O}(g)$ so that it is generated by global sections (while $\mathcal{K}_1(g-1)$ is not) we get a sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}^{\text{rank}\,\mathcal{K}_1 - 2} \to \mathcal{K}_1(g) \to \mathcal{E}(g) \to 0,$$

where $\mathcal{E}$ is a 2-bundle which seems to always be semi-stable and to split generically as $\mathcal{O}(a) \bigoplus \mathcal{O}(a)$ or $\mathcal{O}(a) \bigoplus \mathcal{O}(a-1)$. Semi-stability in turn gives better vanishing estimates (see Hartshorne [12]) than the ones in [9]. In short, we believe there is a deep relationship between semi-stability and the "$2r+1$" conjecture. We illustrate this with a number of interesting examples (all calculations were made using Macaulay II [10]).

Example 2.4. Let Δ be the simplicial complex pictured below.



When $r = 1$, Theorem 2.3 applies and we find that $H^1(\mathcal{K}(d)) = 0$ when $d \geq 4$, which equals the Alfeld–Schumaker bound of $3r+1$. Now consider the case when $r = 2$. We have the usual exact sequence:

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbf{P}^2}^9(-3) \to \mathcal{O}_{\mathbf{P}^2}^2 \to \mathcal{C} \to 0.$$

Here $\mathcal{C}$ is a skyscraper sheaf supported at the two interior vertices with stalks isomorphic to $\mathbf{C}^7$. Our $\mathcal{K}$ has rank 7 and $c_1(\mathcal{K}) = -27$. Via computation we find:

(1) $\mathcal{K} \cong \mathcal{O}_{\mathbf{P}^2}^4(-3) \bigoplus \mathcal{K}_1$ with $\mathcal{K}_1$ indecomposable;
(2) $\mathcal{K}_1|_L \cong \mathcal{O}_L^3(-5)$ if L is a generic line;
(3) $0 \to \mathcal{O}_{\mathbf{P}^2}(-7) \to \mathcal{O}_{\mathbf{P}^2}^2(-5) \bigoplus \mathcal{O}_{\mathbf{P}^2}^2(-6) \to \mathcal{K}_1 \to 0$ is a resolution of $\mathcal{K}_1$;
(4) $\chi(\mathcal{K}(d)) = \frac{7}{2}d^2 - \frac{33}{2}d + 21$;

(5) $c_1(\mathcal{K}_1) = -15 \quad c_2(\mathcal{K}_1) = 76 \quad b(\mathcal{K}_1) = -5$;

(6) $\delta(\mathcal{K}_1) = 1 \quad \left(\delta = c_2 - \sum_{i<j} b_i b_j\right)$, where $\mathcal{K}_1|_L \simeq \bigoplus \mathcal{O}_L(b_i)$, see (2) above.

From (3), we see that $H^2(\mathcal{K}(d)) = 0$ for $d \geq 4$ (which is equal to $2r$), and by Elencwajg and Forster $H^1(\mathcal{K}(d)) = 0$ for $d \geq \delta - b - 1 = 5$ (which is $2r + 1$). Also note that $\dim H^1(\mathcal{K}((4)) = 1$. We have:

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-2) \to \mathcal{O}^2_{\mathbf{P}^2} \bigoplus \mathcal{O}^2_{\mathbf{P}^2}(-1) \to \mathcal{K}_1^{\text{norm}} \to 0,$$

where

(7) $\mathcal{K}_1^{\text{norm}} = \mathcal{K}_1(5)$;

(8) $\mathcal{K}_1^{\text{norm}}|_L \cong \mathcal{O}_L^3$ for L a generic line, so the generic splitting type of $\mathcal{K}_1^{\text{norm}}$ is (0,0,0);

(9) $c_1(\mathcal{K}_1^{\text{norm}}) = 0, \quad c_2(\mathcal{K}_1^{\text{norm}}) = 1, \quad \delta(\mathcal{K}_1^{\text{norm}}) = 1, \quad b(\mathcal{K}_1^{\text{norm}}) = 0$ and $H^1(\mathcal{K}_1^{\text{norm}}(d)) = 0$ for $d \geq 0$.

By Theorem 1.2 of Elencwajg and Forster $\mathcal{K}_1(6)$ is generated by global sections, which we can also see from (3); $\mathcal{K}_1(5)$ is not generated by global sections. Thus by Serre's theorem we obtain a two bundle $\mathcal{E}$ and an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2} \to \mathcal{K}_1(6) \to \mathcal{E}(6) \to 0,$$

where

$$c_1(\mathcal{K}_1^{\text{norm}}(1)) = c_1(\mathcal{K}_1(6)) = 3, \qquad c_1(\mathcal{K}_1^{\text{norm}}(-1)) = -3,$$
$$c_2(\mathcal{K}_1^{\text{norm}}(1)) = c_2(\mathcal{K}_1(6)) = 4\ , \qquad c_2(\mathcal{K}_1^{\text{norm}}(-1)) = 4.$$

Tensoring with $\mathcal{O}_{\mathbf{P}^2}(-2)$ yields

(10) $$0 \to \mathcal{O}_{\mathbf{P}^2}(-2) \to \begin{array}{c}\mathcal{K}_1(4)\\ \| \\ \mathcal{K}_1^{\text{norm}}(-1)\end{array} \to \begin{array}{c}\mathcal{E}(4)\\ \| \\ \mathcal{E}^{\text{norm}}\end{array} \to 0,$$

where $\mathcal{E}(4) = \mathcal{E}^{\text{norm}}$, $c_1(\mathcal{E}(4)) = c_1(\mathcal{E}^{\text{norm}}) = -1$, $c_2(\mathcal{E}(4)) = c_2(\mathcal{E}^{\text{norm}}) = 2$.

For a generic line L, $\mathcal{E}^{\text{norm}}|_L \cong \mathcal{O}_L \bigoplus \mathcal{O}_L(-1)$, which follows by restricting (10) to L:

$$0 \to \mathcal{O}_L(-2) \to \mathcal{O}_L^3(-1) \to \mathcal{O}_L(a) \bigoplus \mathcal{O}_L(-a-1) \to 0.$$

If $a > 0$ we get a contradiction via

$$0 \to H^0\Big(\mathcal{O}_L(a) \bigoplus \mathcal{O}_L(-a-1)\Big) \to H^1(\mathcal{O}_L(-2)) \to 0$$

as $\dim H^1(\mathcal{O}_L(-2)) = 1$. Thus $a = 0$. Since $c_1(\mathcal{E}^{\text{norm}})$ is odd, to show $\mathcal{E}^{\text{norm}}$ is stable it suffices to show $H^0(\mathcal{E}^{\text{norm}}) = 0$ (see [14], p. 165). From (10) we have

$$\ldots \to H^0(\mathcal{K}_1(4)) \to H^0(\mathcal{E}^{\text{norm}}) \to 0 \to \ldots$$

and from (3) $H^0(\mathcal{K}_1(4)) = 0$. Thus $\mathcal{E}^{\text{norm}}$ is stable. The moduli $\mathcal{M}_{\mathbf{P}^2}(-1,2)$ of stable 2 bundles on $\mathbf{P}^2$ with $c_1 = -1$ and $c_2 = 2$ is described in [14]. The jump locus of $\mathcal{E}^{\text{norm}}$ need not be a curve, but the jump lines of the second kind are a curve of degree $2(c_2 - 1) = 2$, and jumping lines are contained in the singular locus of the curve. When L is one of the three lines $\{x = 0\}$, $\{x = 2z\}$, $\{x = 4z\}$, a computation shows that the splitting type of $\mathcal{K}_1$ is:

$$\mathcal{O}_L(-4) \bigoplus \mathcal{O}_L(-5) \bigoplus \mathcal{O}_L(-6).$$

In $\mathbf{P}^{2^\vee}$ the corresponding three points line on the line $\{y = 0\}$, so the set of jump lines of the second kind is the double line $\{y^2 = 0\} \subseteq \mathbf{P}^{2^\vee}$.

Example 2.5. Δ is the same simplicial complex as in Example 2.4, but this time we take $r = 3$.

We now have an exact sequence of the form

$$0 \to \mathcal{K} \to \mathcal{O}^9_{\mathbf{P}^2}(-4) \to \mathcal{O}^2_{\mathbf{P}^2} \to \mathcal{C} \to 0.$$

We compute that $\mathcal{K} \cong \mathcal{O}^4_{\mathbf{P}^2}(-4) \bigoplus \mathcal{K}_1$ with $\mathcal{K}_1$ an indecomposable 3-bundle on $\mathbf{P}^2$; $\mathcal{K}_1$ has a resolution

$$0 \to \mathcal{O}^2_{\mathbf{P}^2}(-9) \to \mathcal{O}^3_{\mathbf{P}^2}(-8) \bigoplus \mathcal{O}^2_{\mathbf{P}^2}(-7) \to \mathcal{K}_1 \to 0.$$

Tensoring this sequence with $\mathcal{O}_{\mathbf{P}^2}(8)$ gives

$$0 \to \mathcal{O}^2_{\mathbf{P}^2}(-1) \to \mathcal{O}^3_{\mathbf{P}^2} \bigoplus \mathcal{O}^2_{\mathbf{P}^2}(1) \to \mathcal{K}_1(8) \to 0.$$

From this sequence we see that $\mathcal{K}_1(8)$ is generated by global sections, while $\mathcal{K}_1(7)$ is not. Also $c_1(\mathcal{K}_1(8)) = 4$ and $c_2(\mathcal{K}_1(8)) = 8$. As in the previous example, Serre's Theorem yields an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2} \to \mathcal{K}_1(8) \to \mathcal{E}(8) \to 0.$$

If we normalize then $\mathcal{E}^{\text{norm}} = \mathcal{E}(6)$ and $c_1(\mathcal{E}^{\text{norm}}) = 0$ and $c_2(\mathcal{E}^{\text{norm}}) = 4$. Notice that $H^1(\mathcal{K}_1(d)) = 0$ for $d \geq 7$ (which is $2r + 1$), but $\dim H^1(\mathcal{K}_1(6)) = 1$. It follows that $\dim H^1(\mathcal{E}^{\text{norm}}) = 1$, so that $\mathcal{E}^{\text{norm}}$ is indecomposable. Moreover $H^0(\mathcal{K}_1(6)) \cong 0$ so $H^0(\mathcal{E}^{\text{norm}}) = 0$ and $\mathcal{E}^{\text{norm}}$ is therefore a stable 2-bundle on $\mathbf{P}^2$. Calculations show that for a generic L the splitting type of $\mathcal{K}_1|_L$ is

$$\mathcal{O}_L(-6) \bigoplus \mathcal{O}^2_L(-7).$$

Since we have

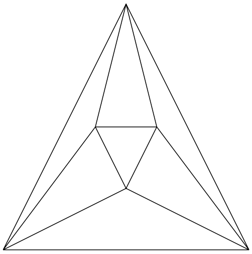
$$0 \to \mathcal{O}_L(-2) \to \mathcal{K}_1(6)|_L \to \mathcal{E}^{\text{norm}}|_L \to 0$$

we see that $\dim H^0(\mathcal{E}^{\text{norm}}|_L) = 2$ and that $\mathcal{E}^{\text{norm}}$ splits as $\mathcal{O}_L \bigoplus \mathcal{O}_L$ on a generic line (recall $\mathcal{E}^{\text{norm}}$ is stable). The cohomology vanishing estimates of Elencwajg and Forster show that for $d \geq 9$

$$H^1(\mathcal{K}_1(d)) = 0$$

(here $\delta(\mathcal{K}_1(8)) = 3$ and $b(\mathcal{K}_1(8)) = 1$). This is greater than the conjectured value $2r + 1$. Notice that the Alfeld–Schumaker bound of $3r + 1$ yields $H^1(\mathcal{K}(d)) = 0$ if $d \geq 10$, whereas Theorem 2.3 gives vanishing for $d \geq 11$.

Example 2.6. In our next example, we take $r = 1$ for Δ as below:



We have

$$0 \to \mathcal{K} \to \mathcal{O}^9_{\mathbf{P}^2}(-2) \to \mathcal{O}^3_{\mathbf{P}^2} \to \mathcal{C} \to 0.$$

Calculation shows $\mathcal{K} \cong \mathcal{O}_{\mathbf{P}^2}(-2) \bigoplus \mathcal{O}^2_{\mathbf{P}^2}(-3) \bigoplus \mathcal{K}_1$ where $\mathcal{K}_1$ is an indecomposable rank 3 bundle on $\mathbf{P}^2$. $\mathcal{K}_1$ has a resolution

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-5) \to \mathcal{O}_{\mathbf{P}^2}(-3) \bigoplus \mathcal{O}^3_{\mathbf{P}^2}(-4) \to \mathcal{K}_1 \to 0$$

and generic splitting type $(-3, -3, -4)$. $\mathcal{K}(4)$ is generated by global sections so by Serre's theorem we get a 2-bundle $\mathcal{E}$ and an exact sequence

$$\begin{array}{ccccccccc} 0 \to & \mathcal{O}_{\mathbf{P}^2}(-1) & \to & K_1(3) & \to & \mathcal{E}(3) & \to 0. \\ &&&&& \| & \\ &&&&& \mathcal{E}^{\text{norm}} & \end{array}$$

Here $\mathcal{E}^{\text{norm}} = \mathcal{E}(3)$ has $c_1(\mathcal{E}^{\text{norm}}) = 0$, $c_2(\mathcal{E}^{\text{norm}}) = 1$. Also $\dim H^0(\mathcal{E}^{\text{norm}}) = 1$ while $H^0(\mathcal{E}^{\text{norm}}(-1)) = 0$. From this we see $\mathcal{E}^{\text{norm}}$ is semi-stable. It is the restriction of the null correlation bundle on $\mathbf{P}^3$ to $\mathbf{P}^2$.

Example 2.7. Let $r = 2$ and let Δ be combinatorially equivalent to Example 2.6, but perturbed so that the three lines which result from joining a boundary vertex to the interior vertex furthest from it do not meet in a point (i.e. the "generic" version of Example 2.6). A computation shows that $\mathcal{K}$ fits into the exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^9_{\mathbf{P}^2}(-4) \longrightarrow \mathcal{O}^3_{\mathbf{P}^2}(-3) \longrightarrow 0,$$

so the worst possible splitting type allowed is

$$\mathcal{K}|_L \simeq \mathcal{O}^5_L(-4) \bigoplus \mathcal{O}_L(-7).$$

Thus, $\delta(\mathcal{K}) = 6$, $b(\mathcal{K}) = -7$, hence $H^1(\mathcal{K}(d)) = 0$ if $d \geq 12$. The Alfeld–Schumaker bound for this example is $H^1(\mathcal{K}(d)) = 0$ if $d \geq 7$. However, for this example, $\mathcal{K}$ splits as evenly as possible, i.e.

$$\mathcal{K}|_L \simeq \mathcal{O}^3_L(-4) \bigoplus \mathcal{O}^3_L(-5).$$

By taking this into consideration, we obtain the same bound as Alfeld–Schumaker.

In these and many other examples, the indecomposable summands of $\mathcal{K}$ split as evenly as possible, and so we conjecture:

Conjecture 2.8. Suppose $\mathcal{K} \simeq \bigoplus \mathcal{K}_i$, with $\mathcal{K}_i$ indecomposable. Then $e(\mathcal{K}_i) \leq 1$ $\forall i$.

In particular, if

$$\mathcal{K} \cong \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^2}(a_i) \bigoplus \mathcal{K}_1 \quad a_1 \geq a_2 \geq \ldots \geq a_\ell,$$

with $\mathcal{K}_1$ indecomposable, then $\mathcal{K}_1|_L \cong \mathcal{O}_L(b_1) \bigoplus \cdots \bigoplus \mathcal{O}_L(b_m)$ with $b_1 \geq \ldots \geq b_m$ where $\ell + m = f_1^0 - f_0^0 = f_2 - 1$ and $0 \leq b_1 - b_m \leq 1$. Moreover we have $a_1 - b_m \leq r$ and $a_\ell \geq b_1$. In the examples we have computed, the associated two bundle $\mathcal{E}$ that we obtain is semistable, and the "$2r + 1$" conjecture is consistent with the improved vanishing expected if $\mathcal{E}$ is semistable.

3. Conditions for $\mathcal{K}$ to be a bundle

In [16], the lower homology modules of the complex $\mathcal{R}/\mathcal{J}$ were analyzed. There are two main results. First, if $i < n$, then $H_i(\mathcal{R}/\mathcal{J})$ has dimension at most $i-1$. (Recall that the dimension of an R module M is the dimension of the ring $R/ann(M)$, in particular, a zero dimensional object is Artinian). Second, the modules $H_i(\mathcal{R}/\mathcal{J})$, $i < n$ all vanish iff $C^r(\hat{\Delta})$ is free. The previous result generalizes to the case of bundles:

Theorem 3.1. *$\mathcal{K}$ is a bundle iff $H_i(\mathcal{R}/\mathcal{J})$ is zero-dimensional, for all $i < n$.*

The proof is very close to the proof of Theorem 4.10 of [16], so we will be brief. Take a Cartan-Eilenberg resolution for the complex $\mathcal{R}/\mathcal{J}$, and dualize. We obtain a spectral sequence; and for the vertical filtration, the $E^1_{i,j}$ terms are given by:

$$E^1_{i,j} = \begin{cases} \mathrm{Ext}^j\left(\bigoplus_{\beta \in \Delta_i^0} R/J(\beta), R\right) & \text{for } i + j = n \\ 0 & \text{otherwise }. \end{cases}$$

Thus, for the vertical filtration, the E^1 terms are stable, with $E^\infty_{i,j} = E^1_{i,j}$, and these terms vanish if $i + j \neq n$. For the remainder of this section, we will use Ext^j_i to denote $\mathrm{Ext}^j(H_i(\mathcal{R}/\mathcal{J}), R)$.

Lemma 3.2. *If $H_i(\mathcal{R}/\mathcal{J})$ is zero-dimensional for all $i < n$, then $\mathcal{K}$ is a bundle.*

Proof. If the $H_i(\mathcal{R}/\mathcal{J})$ are supported only at the maximal ideal all $i < n$, then the $E^2_{i,j}$ terms for the horizontal filtration are given by

$$\begin{array}{ccccc} \mathrm{Ext}_1^{n+1} & \mathrm{Ext}_2^{n+1} & \mathrm{Ext}_3^{n+1} & \ldots & \mathrm{Ext}_n^{n+1} \\ 0 & 0 & \ldots & 0 & \mathrm{Ext}_n^{n} \\ 0 & \ldots & \ldots & 0 & \vdots \\ 0 & \ldots & \ldots & 0 & \mathrm{Ext}_n^{1} \\ 0 & \ldots & \ldots & 0 & \mathrm{Ext}_n^{0} \end{array}$$

By comparing this to the vertical filtration, we see that the only nonvanishing differentials map $\mathrm{Ext}^i(H_n(\mathcal{R}/\mathcal{J}), R)$ to $\mathrm{Ext}^{n+1}(H_{n-i}(\mathcal{R}/\mathcal{J}))$; since the kernel and cokernel of this map are stable and appear in position (i, j) with $i + j > n$, it follows that these maps are isomorphisms. But this means that all the higher Ext modules of $\mathcal{K}$ are supported only at the maximal ideal, so $\mathcal{K}$ is a bundle (for more details, see [14], Chapter II, Sect. 1). □

Lemma 3.3. *If $\mathcal{K}$ is a bundle, then $H_i(\mathcal{R}/\mathcal{J})$ is zero dimensional, for all $i < n$.*

Proof. If $\mathcal{K}$ is a bundle, then Ext^i_n are supported only at the maximal ideal, for all $i > 0$. $H_i(\mathcal{R}/\mathcal{J})$ is supported in codimension at least $n + 2 - i$, so

$$\mathrm{Ext}^j_i = 0 \text{ if } j \leq n + 1 - i.$$

Thus, the E^2 terms of the horizontal filtration are:

$$\begin{array}{ccccc}
\mathrm{Ext}^{n+1}_1 & \mathrm{Ext}^{n+1}_2 & \dots & \mathrm{Ext}^{n+1}_{n-1} & \mathrm{Ext}^{n+1}_n \\
0 & \mathrm{Ext}^{n}_2 & \dots & \vdots & \vdots \\
\vdots & \vdots & 0 & \mathrm{Ext}^{3}_{n-1} & \mathrm{Ext}^{3}_n \\
\vdots & \vdots & \vdots & 0 & \mathrm{Ext}^{2}_n \\
\vdots & \vdots & \vdots & 0 & \mathrm{Ext}^{1}_n \\
0 & 0 & 0 & 0 & \mathrm{Ext}^{0}_n .
\end{array}$$

If we can force the modules Ext^j_i, $i + j = n + 2$, $j \neq n + 1, 2$, (i.e. modules along the lowest diagonal, except those on the edges) to have support in codimension greater than $n+2-i$, then they will have to vanish, and we'll be done by induction. So, suppose we've forced $\mathrm{Ext}^j_{n+2-j} = 0$ for all $n + 1 \neq j < p$, and consider Ext^p_{n+2-p}. The only differential which ever reaches position $(p, n + 2 - p)$ comes from Ext^1_n, so has image a module of finite length (since $\mathcal{K}$ is a bundle, Ext^j_n is finite length for all $j > 0$). On the other hand, the differentials out of position $(p, n+2-p)$ all map to objects of codimension greater than $n + 2 - p$. So if Ext^p_{n+2-p} has support in codimension $n+2-p$, then $E^\infty_{p,n+2-p}$ will also have to have support in codimension $n+2-p$. But comparing this term to the vertical filtration shows that $E^\infty_{p,n+2-p} = 0$, and hence $E^\infty_{p,n+2-p}$ has support in codimension at least $n+3-p$, in particular, Ext^p_{n+2-p} must vanish. □

Notice that we have actually proved the general theorem mentioned in the introduction:

Theorem 3.4. *Let*

$$\mathcal{C} : 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

be a complex of graded modules on $k[x_0, \dots , x_n]$, with C_i Cohen–Macaulay, supported in codimension $n - i$. If

$$H_i(\mathcal{C}) \text{ is supported in codimension } \geq n + 2 - i, \forall i < n$$

then (the sheaf associated to) $H_n(\mathcal{C})$ is a bundle iff $H_i(\mathcal{C})$ is of finite length for all $i < n$. From [16], $H_n(\mathcal{C})$ splits as a sum of line bundles iff $H_i(\mathcal{C})$ vanishes for all $i < n$.

Corollary 3.5. *If $\mathcal{K}$ is a bundle, then the complex $\mathcal{R}/\mathcal{J}$ is exact as a sequence of sheaves.*

Corollary 3.6. *$\mathcal{K}$ is a bundle iff $\mathrm{Ext}_n^{n-i} \simeq \mathrm{Ext}_{n-i}^{n+1}$ for all $i > 1$. In this case, $\mathcal{K}^\vee$ has a filtration with quotients $\mathrm{Ext}^j(\bigoplus_{\beta\in\Delta_i^0} R/J(\beta), R)$.*

Corollary 3.7. *If $\mathcal{K}$ is a bundle, then the Hilbert polynomial $\chi(\mathcal{K}(d))$ is determined by local data for d sufficiently large, in particular, for $d \geq Q(c_1(\mathcal{K}), c_2(\mathcal{K}), e(\mathcal{K}))$.*

Proof. We have:

$$\begin{aligned} H^i(\mathcal{K}(d)) &\simeq \mathrm{Ext}^{n-i}(H_{n-1}(\mathcal{J}), R)_{-d-n-1} && \text{local duality [8]} \\ &\simeq (\mathrm{Ext}_{n-i}^{n+1})_{-d-n-1} && \text{Corollary 3.5} \\ &\simeq H_{n-i}(\mathcal{R}/\mathcal{J})_d && H_{n-i}(\mathcal{R}/\mathcal{J}) \text{ is finite length} \end{aligned}$$

The general results of Elencwajg–Forster give a bound for the vanishing of the higher cohomology of $\mathcal{K}(d)$, which gives us a degree where the complex $\mathcal{R}/\mathcal{J}$ is exact (at the level of modules), hence a bound where we may compute the spline space dimension from local data. □

Remark 3.8. It is natural to ask if the filtration in Corollary 3.6 is related to the Beilinson spectral sequence. This does not seem to be the case; consider Example 2.7 of the last section. We have an explicit resolution for $\mathcal{K}$, so it is easy to compute the relevant cohomology; there are nonzero E^1 terms only for $E^1_{-2,0}$, $E^1_{-1,0}$, $E^1_{0,0}$. Thus, the spectral sequence has a single nonzero row, so it degenerates, and Beilinson's theorem yields an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}^{45}(-1) \longrightarrow \Omega_{\mathbf{P}^2}^{72}(1) \longrightarrow \mathcal{O}_{\mathbf{P}^2}^{105} \longrightarrow \mathcal{K}^\vee \longrightarrow 0.$$

In particular, the filtration we obtain from the Beilinson spectral sequence is just $\mathcal{K}^\vee$ itself.

4. The general case

Let Δ be an n-dimensional simplicial complex, which triangulates a ball in k^n. The first point is that the top two Chern classes of $\mathcal{K}$ are essentially the same as in the $\mathbf{P}^2$ case.

Lemma 4.1.

$$\begin{aligned} c_1(\mathcal{K}) &= -f_{n-1}^0(r+1) \\ c_2(\mathcal{K}) &= \binom{f_{n-1}^0}{2}(r+1)^2 - \binom{r+2}{2} f_{n-2}^0 \\ &\quad + \frac{1}{2}\sum_{v\in\Delta_{n-2}^0}((k_v-1)\alpha_v^2 + (k_v-2r-3)\alpha_v). \end{aligned}$$

Proof. It follows from Hirzebruch–Riemann–Roch that if $\mathcal{M}$ is a coherent sheaf on $\mathbf{P}^n$, supported in codimension at least r, then $c_i(\mathcal{M}) = 0$ for $i < r$. By Lemma 3.1 of [16], if $i < n$, then $H_i(\mathcal{R}/\mathcal{J})$ is supported in codimension at least $n+2-i$, and if $i \leq n-3$, then $\bigoplus_{\beta \in \Delta_i^0} R/J(\beta)$ is supported in codimension at least three. Then the result follows from the Whitney product formula. □

The other invariant we need is $e(\mathcal{K})$. A straightforward generalization of Lemma 3.1 of [17], and Lemma 3.1 of [16] yields an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-r-1)^{f_{n-1}^0} \xrightarrow{\Phi} \mathcal{O}_{\mathbf{P}^n}^{f_{n-1}^0 - f_n + 1} \longrightarrow \mathcal{C} \longrightarrow 0,$$

where $\mathcal{C}$ is supported in codimension at least two; in fact, if $\mathcal{K}$ is a bundle then $\mathcal{C} \simeq \mathrm{im}\partial_{n-1}$.

Restricting to a generic line L, it is obvious that $\mathcal{C}|_L = 0$, so we get an exact sequence:

$$0 \longrightarrow \mathcal{K}|_L \longrightarrow \mathcal{O}_L(-r-1)^{f_{n-1}^0} \longrightarrow \mathcal{O}_L^{f_{n-1}^0 - f_n + 1} \longrightarrow 0,$$

and we have a bound on $e(\mathcal{K})$ exactly like that of Sect. 2, i.e.

$$e(\mathcal{K}) \leq |c_1(\mathcal{K}) + (f_n - 1)(r+1)| = |(f_n - f_{n-1}^0 - 1)|(r+1).$$

The proof of Elencwajg and Forster that $H^i(\mathcal{K}(d))$ vanishes for d bigger than a polynomial in c_1, c_2 and e is inductive, and based on the $\mathbf{P}^2$ bound. We close by working out (for our situation) the $\mathbf{P}^3$ case explicitly. We will use that if $\mathcal{E}$ is a rank r bundle on $\mathbf{P}^n$, then

$$c_1(\mathcal{E}(m)) = c_1(\mathcal{E}) + rm$$
$$c_2(\mathcal{E}(m)) = c_2(\mathcal{E}) + m(r-1)c_1(\mathcal{E}) + \binom{r}{2} m^2.$$

We can use induction and the exact sequence

$$0 \longrightarrow \mathcal{K}(d-1) \longrightarrow \mathcal{K}(d) \longrightarrow \mathcal{K}|_{\mathbf{P}^2}(d) \longrightarrow 0$$

to force vanishing of $H^i(\mathcal{K}(d))$, $i \geq 2$; to use the $\mathbf{P}^2$ result we need to know e and b. Since restricting $\mathcal{K}|_{\mathbf{P}^2}$ to a generic line is the same thing as restricting $\mathcal{K}$ to a generic line, we already have worst case values for e and b;

$$e(\mathcal{K}|_{\mathbf{P}^2}) \leq |(f_3 - f_2^0 - 1)|(r+1), \text{ and}$$
$$b(\mathcal{K}|_{\mathbf{P}^2}) \geq (f_3 - f_2^0 - 2)(r+1).$$

From 2.6 of [9], we know that

$$\delta(\mathcal{K}|_{\mathbf{P}^2}) \leq c_2(\mathcal{K}|_{\mathbf{P}^2}) - \frac{\mathrm{rk}\,\mathcal{K} - 1}{2\mathrm{rk}\,\mathcal{K}} c_1(\mathcal{K}|_{\mathbf{P}^2})^2 + \frac{\mathrm{rk}\,\mathcal{K}}{8} e(\mathcal{K}|_{\mathbf{P}^2})^2.$$

Thus, $H^i(\mathcal{K}(d)) = 0$, for $i \geq 2$ and

$$d \geq c_2(\mathcal{K}|_{\mathbf{P}^2}) - \frac{f_3 - 2}{2(f_3 - 1)}(f_2^0(r+1))^2 + \frac{f_3 - 1}{8} e(\mathcal{K}|_{\mathbf{P}^2})^2 - b(\mathcal{K}|_{\mathbf{P}^2}) - 1.$$

(Call the above value Q_0; the value for $c_2(\mathcal{K}|_{\mathbf{P}^2})$ comes from Lemma 4.1). Now we just need to determine the vanishing of $H^1(\mathcal{K}(d))$; following the proof of Theorem 3.3 [9] we first need to find Q_1 such that for all $d \geq Q_1$

$$H^1\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{K}(d+1)|_{\mathbf{P}^2}\big) = 0.$$

Consider $\Omega^1_{\mathbf{P}^2} \otimes \mathcal{K}(1)|_{\mathbf{P}^2}$. For a generic line,

$$\Omega^1_{\mathbf{P}^2}|_L \simeq \mathcal{O}_L(-1) \bigoplus \mathcal{O}_L(-2),$$

and twisting $\mathcal{K}$ does not change e, so

$$e\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{K}(1)|_{\mathbf{P}^2}\big) = e\big(\mathcal{K}|_{\mathbf{P}^2}\big) + 1, \text{ and}$$
$$b\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{K}(1)|_{\mathbf{P}^2}\big) = b\big(\mathcal{K}|_{\mathbf{P}^2}\big) - 1.$$

For any bundle $\mathcal{E}$ on $\mathbf{P}^2$, the exact sequence

$$0 \longrightarrow \Omega^1_{\mathbf{P}^2} \otimes \mathcal{E} \longrightarrow \mathcal{E}^3(-1) \longrightarrow \mathcal{E} \longrightarrow 0$$

yields

$$c_1\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{E}\big) = 2c_1(\mathcal{E}) - 3rk(\mathcal{E}), \text{ and}$$
$$c_2\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{E}\big) = c_1(\mathcal{E})^2 + 2c_2(\mathcal{E}) - 6c_1(\mathcal{E})rk(\mathcal{E}) + 3c_1(\mathcal{E}) + 3rk(\mathcal{E}) + 9\binom{rk(\mathcal{E})}{2}.$$

Since

$$c_1\big(\mathcal{K}(1)|_{\mathbf{P}^2}\big) = -f_2^0(r+1) + f_3 - 1, \text{ and}$$
$$c_2\big(\mathcal{K}(1)|_{\mathbf{P}^2}\big) = c_2(\mathcal{K}) + (f_3-2)(-f_2^0(r+1)) + \binom{f_3-1}{2},$$

we can just plug these values in to the previous formula to obtain

$$c_1\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{K}(1)|_{\mathbf{P}^2}\big) = -2f_2^0(r+1) - f_3 + 1,$$

and

$$c_2\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{K}(1)|_{\mathbf{P}^2}\big) = 2c_2(\mathcal{K}) + (f_2^0(r+1))^2 + 2f_3 f_2^0(r+1) + \frac{1}{2}(f_3-1)(f_3-6).$$

Computing, we find that the desired Q_1 is

$$c_2\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{K}(1)|_{\mathbf{P}^2}\big) - \frac{f_3-2}{2(f_3-1)} c_1\big(\Omega^1_{\mathbf{P}^2} \otimes \mathcal{K}(1)|_{\mathbf{P}^2}\big)^2 + \frac{f_3-1}{8}(e(\mathcal{K}|_{\mathbf{P}^2})+1)^2 - b(\mathcal{K}|_{\mathbf{P}^2}).$$

Finally, let k_0 be the max of Q_0, Q_1; applying Theorem 1.2, Lemma 4.1, and the proof of Theorem 3.3 of [9], we obtain a vanishing theorem for the higher cohomology of $\mathcal{K}$ in the $\mathbf{P}^3$ setting.

Theorem 4.2. *Let*

$$k_1 = \delta(\mathcal{K}(k_0)) \cdot (1 + \max\{0, \delta(\mathcal{K}(k_0)) - b(\mathcal{K}(k_0)^*) + 2\}),$$

then

$$H^i(\mathcal{K}(d)) = 0 \ \textit{for all} \ d \geq k_0 + k_1 - 1, i \geq 1.$$

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