Changing coordinates

1. Consider the region $R$ in $\mathbb{R}^2$ shown below at right. In this problem, you will do a change of coordinates to evaluate:

$$\int \int_R x - 2y \, dA$$

(a) Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square $S$ to $R$. Write you answer both as a matrix \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\] and as $T(u, v) = (au + bv, cu + dv)$, and check your answer with the instructor.

(b) Compute $\int \int_R x - 2y \, dA$ by relating it to an integral over $S$ and evaluating that. Check your answer with the instructor.

2. Another simple type of transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a translation, which has the general form $T(u, v) = (u + a, v + b)$ for a fixed $a$ and $b$.

(a) If $T$ is a translation, what is its Jacobian matrix? How does it distort area?

(b) Consider the region $S = \{u^2 + v^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinates $(u, v)$, and the region $R = \{(x - 2)^2 + (y - 1)^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinates $(x, y)$.

Make separate sketches of $S$ and $R$.

(c) Find a translation $T$ where $T(S) = R$.

(d) Use $T$ to reduce

$$\int \int_R x \, dA$$

to an integral over $S$, and then evaluate that new integral using polar coordinates.

(e) Check your answer in (d) with the instructor.

Problems 3 and 4 on the back.
3. Consider the region $R$ shown below. Here the curved left side is given by $x = y - y^2$. In this problem, you will find a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square $S = [0,1] \times [0,1]$ to $R$.

(a) As a warm up, find a transformation that takes $S$ to the rectangle $[0,2] \times [0,1]$ which contains $R$.

(b) Returning to the problem of finding $T$ taking $S$ to $R$, come up with formulas for $T(u,0)$, $T(u,1)$, $T(0,v)$, and $T(1,v)$. Hint: For three of these, use your answer in part (a).

(c) Now extend your answer in (b) to the needed transformation $T$. Hint: Try “filling in” between $T(0,v)$ and $T(1,v)$ with a straight line.

(d) Compute the area of $R$ in two ways, once using $T$ to change coordinates and once directly.

4. If you get this far, evaluate the integrals in Problems 1 and 2 directly, without doing a change of coordinates. It’s a fun-filled task...