1. Evaluate the following integral by reversing the order of integration:

\[ \int_0^1 \int_{\sqrt[3]{y}}^{1} \sqrt{x^3 + 1} \, dx \, dy. \]

(Hint: When you change to \( dx \, dy \), be sure to also change the bounds of integration.)

**SOLUTION:**

We are integrating over the region below:

Changing the order of integration we get

\[ \int_0^1 \int_{\sqrt[3]{y}}^{x^2} \sqrt{x^3 + 1} \, dy \, dx = \int_0^{x^2} \int_0^1 \sqrt{x^3 + 1} \, dy \, dx \]

\[ \int_0^1 \int_{x^2}^{1} \sqrt{x^3 + 1} \, dx \, dy = \int_0^{1} x^2 \sqrt{x^3 + 1} \, dx = 2/9[(x^3 + 1)^{3/2}]_0^1 = 2/9(2^{3/2} - 1). \]
2. Consider the region bounded by the curves determined by $-2x + y^2 = 6$ and $-x + y = -1$.

(a) Sketch the region $R$ in the plane.

**SOLUTION:**

(b) Set up and evaluate an integral of the form $\iint_R dA$ that calculates the area of $R$.

**SOLUTION:**

$$
\int_{-2}^{4} \int_{y^2-6}^{y+1} dx \, dy = \int_{-2}^{4} \left( y + 1 - \frac{y^2 - 6}{2} \right) dy = \left[ -\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y \right]_{-2}^{4} = 18
$$
3. Consider the region $R$ in the first quadrant which lies above the $x$-axis and between the circles of radius 1 and 2 centered at $(0, 0)$. Without using polar coordinates, evaluate

$$\iint_R y\,dA.$$ 

**SOLUTION:** Notice that both the function $y$ and the region $R$ are symmetric about the $y$-axis, so we can integrate $y$ over the half of $R$ which lies in the first quadrant (Call this $R'$) and double our answer. $R'$ is shown below.

![Diagram of region R']

Break up $R'$ into two parts $A$ and $B$ as above. Integrating, we obtain

$$\iint_{R'} y\,dA = \int_A y\,dA + \int_B y\,dA = \int_0^1 \int_{\sqrt{4-x^2}}^0 y\,dy\,dx + \int_1^2 \int_{\sqrt{4-x^2}}^0 y\,dy\,dx$$

$$= \int_0^1 \frac{y^2}{2} \sqrt{4-x^2} \,dx + \int_1^2 \frac{y^2}{2} \sqrt{4-x^2} \,dx = \int_0^1 \frac{3}{2} \,dx + \int_1^2 \frac{1}{2(4-x^2)} \,dx$$

$$= \frac{7}{3}$$

Now double this value to get $14/3$, which is the integral over the entire region $R$. 
4. Evaluate
\[ \int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx. \]

Hint: don't do it directly.

**SOLUTION:**

The region over which we are integrating is:

Converting to polar we get

\[ \int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx = \int_{\pi/2}^{\pi} \int_{0}^{2} (r^2) r \, dr \, d\theta = 2\pi \]
5. The function \( P(x) = e^{-x^2} \) is fundamental in probability.

(a) Sketch the graph of \( P(x) \). Explain why it is called a “bell curve.”

SOLUTION:

(b) Compute \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \) using the following brilliant strategy of Gauss.

i. Instead of computing \( I \), compute \( I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right) \).

ii. Rewrite \( I^2 \) as an integral of the form \( \iint_R f(x, y) \, dA \) where \( R \) is the entire Cartesian plane.

SOLUTION:

\[
I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2} \, dy \, dx
\]

iii. Convert that integral to polar coordinates.

SOLUTION:

\[
I^2 = \int_{0}^{2\pi} \int_{0}^{\infty} r^2 e^{-r^2} \, dr \, d\theta
\]

iv. Evaluate to find \( I^2 \). Deduce the value of \( I \).

SOLUTION:

\[
\int_{0}^{2\pi} \int_{0}^{\infty} r^2 e^{-r^2} \, dr \, d\theta = 2\pi \oint_{0}^{\infty} r e^{-r^2} \, dr = 2\pi \lim_{t \to \infty} \int_{0}^{t} r e^{-r^2} \, dr = 2\pi \lim_{t \to \infty} \left[ -\frac{1}{2} e^{-t^2} \right]_{0}^{t}
\]

\[
= \pi \lim_{t \to \infty} (-e^{-t^2} + 1) = \pi
\]

So \( I = \sqrt{\pi} \).
6. Compute \[ \int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy. \]

**SOLUTION:**

As in the previous problem, let's convert to polar coordinates.

\[ \int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy = \int_0^{\pi/2} \int_0^\infty \frac{r}{(1 + r^2)^2} \, dr \, d\theta = \frac{\pi}{2} \int_0^\infty \frac{r}{(1 + r^2)^2} \, dr \]

This is an improper integral, so

\[ \frac{\pi}{2} \int_0^\infty \frac{r}{(1 + r^2)^2} \, dr = \frac{\pi}{2} \lim_{t \to \infty} \int_0^t \frac{r}{(1 + r^2)^2} \, dr = \frac{\pi}{4} \lim_{t \to \infty} \left[ -\frac{1}{1 + r^2} \right]_0^t \]

\[ = \frac{\pi}{4} \lim_{t \to \infty} \left( -\frac{1}{1 + t^2} + 1 \right) = \frac{\pi}{4} \]