1. Consider the vector field $\mathbf{F} = (y, 0)$ on $\mathbb{R}^2$.

(a) Draw a sketch of $\mathbf{F}$ on the region where $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$. Check your answer with the instructor.

**SOLUTION:**
Below is the image for parts (a) and (b)

(b) Consider the following two curves which start at $A = (-2, 0)$ and end at $B = (2, 0)$, namely the line segment $C_1$ and upper semicircle $C_2$.
Add these curves to your sketch, and compute both $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Check your answers with the instructor.

**SOLUTION:**
Parametrize $C_1$ by $\mathbf{r}_1(t) = (t, 0), -2 \leq t \leq 2$ and parametrize $C_2$ by $\mathbf{r}_2(t) = (-2 \cos t, 2 \sin t), 0 \leq t \leq \pi$.
We have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^{2} F(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t) \, dt = \int_{0}^{2} (0, 0) \cdot (1, 0) \, dt = 0$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{\pi} F(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t) \, dt = \int_{0}^{\pi} (2 \sin t, 0) \cdot (2 \sin t, 2 \cos t) \, dt = 4 \int_{0}^{\pi} \sin^2(t) \, dt$$

$$= 4 \cdot \frac{1}{2} \left[ t - \frac{1}{2} \sin(2t) \right]_{0}^{\pi} = 2\pi$$

(c) Based on your answer in (b), could $\mathbf{F}$ be $\nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}$? Explain why or why not.

**SOLUTION:**
By the Fundamental Theorem of Line Integrals, if $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}$ then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent for any curve $C$ starting at $A = (-2, 0)$ and ending at $B = (2, 0)$. Since we obtained different answers for the paths $C_1$ and $C_2$, $\mathbf{F}$ cannot be of this form.
2. Consider the curve $C$ and vector field $\mathbf{F}$ shown below.

(a) Calculate $\mathbf{F} \cdot \mathbf{T}$, where here $\mathbf{T}$ is the unit tangent vector along $C$. Without parameterizing $C$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by using the fact that it is equal to $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$.

**SOLUTION:**

From the picture we suppose that $\mathbf{F}(x, y) = (1, 1)$. We have $\mathbf{T} = \frac{1}{\sqrt{5}}(-2, -1)$, so $\mathbf{F} \cdot \mathbf{T} = \frac{3}{\sqrt{5}}$. So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \frac{3}{\sqrt{5}} \int_C ds = -3$$

since $\int_C ds$ is simply the distance between $(1,1)$ and $(3,2)$.

(b) Find a parameterization of $C$ and a formula for $\mathbf{F}$. Use them to check your answer in (a) by computing $\int_C \mathbf{F} \cdot d\mathbf{r}$ explicitly.

**SOLUTION:**

Parametrize $C$ by $\mathbf{r}(t) = (3 - 2t, 2 - t), 0 \leq t \leq 1$. We already have $\mathbf{F} = (1, 1)$. So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (1,1) \cdot (-2,-1) \, dt = -3$$

3. Consider the points $A = (0,0)$ and $B = (\pi,-2)$. Suppose an object of mass $m$ moves from $A$ to $B$ and experiences the constant force $\mathbf{F} = -mg\mathbf{j}$, where $g$ is the gravitational constant.

(a) If the object follows the straight line from $A$ to $B$, calculate the work $W$ done by gravity using the formula from the first week of class.

**SOLUTION:**

Recall that the work done on an object moving along a straight line subject to a constant force $\mathbf{F}$ is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D}$ is the displacement vector. In this case $\mathbf{D} = (\pi,-2)$ and $\mathbf{F} = (0,-mg)$. So $W = (\pi,-2) \cdot (0,-mg) = 2mg$. 

(b) Now suppose the object follows half of an inverted cycloid \( C \) as shown below. Explicitly parameterize \( C \) and use that to calculate the work done via a line integral.

![Image of inverted cycloid](image)

**SOLUTION:**

A parametrization for the inverted cycloid \( C \) is \( \mathbf{r}(t) = (t - \sin t, \cos t - 1), 0 \leq t \leq \pi \). So

\[
W = \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{\pi} (0, -mg) \cdot (1 - \cos t, -\sin t) \, dt = \int_{0}^{\pi} mg \sin t \, dt = mg [-\cos t]_{0}^{\pi} = 2mg
\]

(c) Find a function \( f : \mathbb{R}^2 \to \mathbb{R} \) so that \( \nabla f = \mathbf{F} \). Use the Fundamental Theorem of Line Integrals to check your answers for (a) and (b). Have you seen the quantity \(-f\) anywhere before? If so, what was its name?

**SOLUTION:**

If such an \( f \) exists, we must have \( f_x = 0 \) and \( f_y = -mg \). Integrating \(-mg\) with respect to \( y \) we obtain \( f = -mg y + C(x) \), where \( C(x) \) is some function of \( x \). Differentiating this with respect to \( x \) we obtain \( f_x = C'(x) = 0 \), so \( f = -mg y + K \), where \( K \) is a constant, is a potential function for \( \mathbf{F} \).

By the Fundamental Theorem of Line integrals, both (a) and (b) must have the same answer, namely

\[
\int_{L} \mathbf{F} \cdot d\mathbf{r} = \int_{L} \nabla(f) \cdot d\mathbf{r} = f(B) - f(A) = f(\pi, -2) - f(0, 0) = (-mg(-2) + K) - K = 2mg
\]

where \( L \) is the line segment from \( A \) to \( B \) and

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \nabla(f) \cdot d\mathbf{r} = f(B) - f(A) = 2mg
\]

The quantity \(-f\) is called the *potential energy*. 
48. Experiments show that a steady current $I$ in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). **Ampère’s Law** relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where $I$ is the net current that passes through any surface bounded by a closed curve $C$, and $\mu_0$ is a constant called the permeability of free space. By taking $C$ to be a circle with radius $r$, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance $r$ from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

**SOLUTION:**

We are assuming that $\mathbf{B}$ has magnitude which only depends on the distance from the wire. So $B = |\mathbf{B}|$ is constant along any circle centered around the wire in a plane perpendicular to the wire. Let $C = r(t)$ be such a circle with radius $r$ parametrized in the counterclockwise direction and let $B$ denote the magnitude of $\mathbf{B}$ along $C$. Note that $\mathbf{B}(r(t))$ is a positive multiple of $r'(t)$ by definition. So it follows that $\mathbf{T}(t)$, the unit tangent vector to $C$, is given by $\mathbf{T}(t) = \frac{\mathbf{B}(r(t))}{B}$. We have

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_C \mathbf{B} \cdot \mathbf{T} d\mathbf{s} = \int_C \frac{\mathbf{B} \cdot \mathbf{B}}{B} d\mathbf{s} = B \int_C d\mathbf{s} = 2\pi r B$$

By Ampere's Law, $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$, so we have $2\pi r B = \mu_0 I$, or $B = \frac{\mu_0 I}{2\pi r}$. 