Thursday, October 1  *  Solutions  *  Constrained min/max via Lagrange multipliers.

1. Let $C$ be the curve in $\mathbb{R}^2$ given by $x^3 + y^3 = 16$.

   (a) Sketch the curve $C$.

   **SOLUTION:**

   ![Curve Sketch](image)

   (b) Is $C$ bounded?

   **SOLUTION:**

   No. Given arbitrarily large $y$ values we can find an $x$ value which satisfies the equation. To see this notice that $y = \sqrt[3]{16 - x^3}$, so we can input arbitrarily large (or small) $x$ values and get a $y$ value for that input.

   (c) Is $C$ closed?

   **SOLUTION:**

   Yes, $C$ is closed in $\mathbb{R}^2$.

2. Consider the function $f(x, y) = e^{xy}$ on $C$.

   (a) Is $f$ continuous? What does the Extreme Value Theorem tell you about the existence of global min and max of $f$ on $C$?

   **SOLUTION:**

   Yes, $f$ is continuous. Since $C$ is not bounded, the Extreme Value Theorem does not tell you anything about the existence of a global min and max of $f$ on $C$.

   (b) Use Lagrange multipliers to determine both the min and max values of $f$ on $C$.

   **SOLUTION:**

   Let $g(x, y) = x^3 + y^3$. Our constraint is $g(x, y) = 16$. $\nabla f = (ye^{xy}, xe^{xy})$ and $\nabla g = (3x^2, 3y^2)$, so using the method of Lagrange multipliers we need to find simultaneous solutions in $x$ and $y$ of the following three equations:

   \[
   \begin{align*}
   x^3 + y^3 &= 16 \\
   ye^{xy} &= \lambda 3x^2 \\
   xe^{xy} &= \lambda 3y^2
   \end{align*}
   \]
Multiplying (2) by \(x\) gives \(xye^{xy} = \lambda x^3\) and multiplying (3) by \(y\) gives \(yxe^{xy} = \lambda y^3\). So we have that \(\lambda x^3 = \lambda y^3\). This is satisfied if \(\lambda = 0\) or if \(x^3 = y^3\). If \(\lambda = 0\) we deduce from (2) that \(y = 0\) and from (3) that \(x = 0\). But the point \((0,0)\) is not on the curve \(x^3 + y^3 = 16\), so \(\lambda \neq 0\). So we must have \(x^3 = y^3\), or \(x = y\). Using (1) this implies that \(2x^3 = 16\) or \(x = y = \sqrt[3]{2}\). So \(f\) attains either a maximum or a minimum of \(f(2,2) = e^4\) at \((2,2)\).

I claim \(f(2,2) = e^4\) is the global maximum of \(f\) on \(C\). One way to see this is that since \(f\) has only one critical point on \(C\), it must behave in one of exactly two ways:

i. \(f\) increases on \(C\) as \(x\) increases until it hits \(x = 2\), then \(f\) decreases. In this case \(f\) has a global maximum at \((2,2)\).

ii. \(f\) decreases on \(C\) as \(x\) increases until it hits \(x = 2\), then \(f\) increases. In this case \(f\) has a global minimum at \((2,2)\).

From the graph of \(x^3 + y^3 = 16\) we see that most of \(C\) lies in either the second or fourth quadrant, implying that \(xy < 0\) on most of \(C\), or \(e^{xy} < 1\). Since \(e^4 > 1\), we see that \(f\) cannot have a global minimum at \((2,2)\), so it must have a global maximum there. Since there is no other critical point, \(f\) does not have a minimum on \(C\). In fact we can make \(f\) arbitrarily close to 0 by taking points on \(C\) with either very large or very small \(x\) coordinate.

3. Consider the surface \(S\) given by \(z^2 = x^2 + y^2\)

(a) Sketch \(S\).

**SOLUTION:**

(b) Use Lagrange multipliers to find the points on \(S\) that are closest to \((4, 2, 0)\).

**SOLUTION:**

Minimize the square of the distance function \(D = (x - 4)^2 + (y - 2)^2 + z^2\) from the point \((4, 2, 0)\) subject to the constraint \(g = x^2 + y^2 - z^2 = 0\). We have \(\nabla D = \langle 2(x - 4), 2(y - 2), 2z \rangle\) and \(\nabla g = \langle 2x, 2y, -2z \rangle\). From the picture it is clear that \(D\) attains a global minimum value on \(S\) (i.e. there are points which are closest to \((4, 2, 0)\)). So one of the critical points we find using Lagrange multipliers will correspond to this minimum value and we simply need to evaluate \(D\) at each of the critical points and take the smallest to find the minimum
distance. Using the method of Lagrange multipliers we get the system (divide out by 2 first):

\[
\begin{align*}
(x - 4) &= \lambda x \\
(y - 2) &= \lambda y \\
0 &= -\lambda z
\end{align*}
\]

If \( \lambda \neq -1 \) then \( z = 0 \) from the last equation so the constraining equation \( z^2 = x^2 + y^2 \) implies that \( x = y = 0 \). If \( \lambda = -1 \) then the top two equations give \( x = 2 \) and \( y = 1 \). So the three possible points of minimum distance from \((4, 2, 0)\) are \((0, 0, 0)\), \((2, 1, \sqrt{5})\), and \((2, 1, -\sqrt{5})\). By calculation we see that the squares of the distances of each of these from \((4, 2, 0)\) are 20, 10, and 10 respectively. So the two points \((2, 1, \sqrt{5})\) and \((2, 1, -\sqrt{5})\) on the cone \( z^2 = x^2 + y^2 \) are of minimum distance from the point \((4, 2, 0)\).

4. For the function shown on the back of the sheet, use the level curves to find the locations and types (min/max/saddle) for all the critical points of the function:

\[ f(x, y) = 3x - x^3 - 2y^2 + y^4 \]

Use the formula for \( f \) and the 2nd-derivative test to check your answer.

**SOLUTION:**

Mins and maxes occur where the level curves shrink toward a point and saddle points occur where the level curve intersects itself. From looking at the set of level curves it appears that \( f(x, y) \) has minimums at \((-1, 1)\) and \((-1,-1)\), a maximum at \((1, 0)\), and saddle points at \((-1,0)\), \((1,1)\), and \((1,-1)\).

Now let’s find the critical points precisely. \( f_x = 3(1 - x^2) \) and \( f_y = 4y(y^2 - 1) \). So \( f \) has critical points at \((1,0),(1,1),(-1,0),(-1,1),\) and \((-1,-1)\). \( f_{xx} = -6x, f_{yy} = 12y^2 - 4, \) and \( f_{xy} = 0 \), so the Hessian is \( D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = -6x(12y^2 - 4) \). \( D(-1,0), D(1,1), \) and \( D(1,-1) \) are all negative, so these are saddle points. \( D(1,0), D(-1,1), \) and \( D(-1,-1) \) are all positive so these are maxes and mins. \( f_{xx}(1,0) < 0 \) so \((1,0)\) is a local max. \( f_{xx}(-1,1) \) and \( f_{xx}(-1,-1) \) are both positive so these are local mins. This analysis agrees with our guesses.

5. If the length of the diagonal of a rectangular box must be \( L \), what is the largest possible volume?

**SOLUTION:**

Set \( x \) = length of the box, \( y \) = width of the box, \( z \) = height of the box. This simply supposes that the box is sitting in the octant \( x \geq 0, y \geq 0, \) and \( z \geq 0 \) with its edges along each axis. The volume function is then \( V = xyz \) and the constraint is that \( L^2 = x^2 + y^2 + z^2 \). Using the method of Lagrange multipliers we get the system of equations:
\[yz = 2\lambda x\]
\[xz = 2\lambda y\]
\[xy = 2\lambda z\]
\[x^2 + y^2 + z^2 = L^2\]

Since we want to maximize volume we can assume that \(x > 0, y > 0,\) and \(z > 0\). This rules out the possibility \(\lambda = 0\) (since \(\lambda = 0\) implies at least two of the variables \(x, y,\) and \(z\) are 0). Also this means we can multiply the first equation by \(x\), the second by \(y\), and the third by \(z\) to get a new system:

\[xyz = 2\lambda x^2\]
\[xyz = 2\lambda y^2\]
\[xyz = 2\lambda z^2\]

This implies that \(x^2 = y^2 = z^2\). Coupling this with the constraints \(x > 0, y > 0, z > 0\) we see that this means \(x = y = z\). Plugging this into the constraining equation \(L^2 = x^2 + y^2 + z^2\) we get that \(L^2 = 3x^2\) or \(x = L/\sqrt{3}\). So \(V = (L/\sqrt{3})^3 = L^3/(3\sqrt{3})\) is the biggest possible volume for the box.