Lecture 8

Warm Up

41. Find \[ \lim_{(x,y) \to (0,0)} \frac{e^{-(x^2+y^2)}}{x^2+y^2} - 1 \]

\[ \text{Soln: Polarize:} \quad \lim_{r \to 0} \frac{e^{-r^2}}{r^2} - 1 = \lim_{r \to 0} \frac{-2re^{-r^2}}{2r} \]

and \[ \lim_{r \to 0} -e^{-r^2} = -1 \]

Today 14.3 Partial Differentiation

Def: \[ f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \]

also written as \( \frac{df}{\partial x} \)

Intuition

Slice \( f(x,y) \) with the plane \( y = b \), we see a curve \( g(x) = f(x,b) \), slope of that curve at \( x = a \)!
Of course, can also do with y (in fact, the directional deriv will be for any way of approaching (a,b)).

Example 1 \( f(x,y) = x^3 + x^2 y^3 - 2y^2 \)

Find \( f_x(2,1), f_y(2,1) \)

\[
\begin{align*}
\text{Solu:} & \quad f_x = 3x^2 + 2xy^3, \quad f_x(2,1) = 12 + 4 = 16 \\
\text{} & \quad f_y = 3x^2y^2 - 4y, \quad f_y(2,1) = 8
\end{align*}
\]

This says as you walk towards (2,1) from (2+h,1) the "x" direction, slope is 16, whereas approach along (2,1+h), slope is 8.

Class 1 \( f(x,y) = \sin \left( \frac{x}{1+y} \right) \) compute \( f_x, f_y \)

\[
\begin{align*}
\text{Solu:} & \quad f_x = \cos \left( \frac{x}{1+y} \right) \cdot \frac{1}{1+y} \\
\text{} & \quad f_y = \cos \left( \frac{x}{1+y} \right) \cdot \frac{-x}{(1+y)^2}
\end{align*}
\]

Find \( \frac{\partial z}{\partial x} \) if \( x^3 + y^3 + z^3 + 6xyz = 1 \)

\[
\begin{align*}
\text{Solu implicit} & \quad 3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6y (x \frac{\partial z}{\partial x} + z) = 0 \\
\text{So} & \quad \frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}
\end{align*}
\]
WHO CARES? WHAT'S THE USE?

Let's do a thought experiment. In 1 var, to find min/max of \( y = f(x) \), we looked for \( f'(x) = 0 \), 1-dimensional mountain peak

What do you think will be true on an actual (2d) mountain peak?

\[
e.g. \quad z = f(x,y) = 9 - x^2 - y^2
\]

\[
f_x = -2x
\]

\[
f_y = -2y
\]

\[
f_x = f_y = 0 \quad \text{at} \quad (0,0) \quad \text{mountain top}
\]

Higher Partial: Just like in 1 variable, we could take second derivs, and just like 1 var, they have a meaning in terms of change

Example: for \( f(x,y) = 9 - x^2 - y^2 \), we have

\[
f_{xx} = (f_x)_x = -2 \quad \text{with } y \text{ held const, } f \text{ is concave down}
\]

\[
f_{xy} = (f_x)_y = 0
\]

\[
f_{yx} = (f_y)_x = 0
\]

\[
f_{yy} = (f_y)_y = -2
\]
Suppose we define a function
\[ f(x, y) = z = \sqrt{1 - x^2 + y^2} \]

= top \( \frac{1}{2} \) of picture above.

\[ f_x (0,0) \quad f_{xx} (0, 0) \quad f_y (0, 0) \quad f_{yy} (0, 0) \]

(2)
\[ f_x (0, 2) = 0 \quad f_y (0, 2) + \quad < \]

\[ \text{Solve} \quad (0, 0) \quad \text{both } f_x, f_y \text{ are zero (horizontal tangent plane), } f_{xx} = 0 \quad f_{yy} = 0 \]

(\text{any such is a saddle point})

\[ \text{DEF} \quad \text{Saddle } \iff \quad D = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} < 0 \]
Suppose we're at the origin of a saddle.

- What do you think $f_x(0,0)$ and $f_y(0,0)$ will be?

**HINT**: Do a King Solomon to Mr. Ed.

- What about the (gasp!) second partials?