Chapter 12.1

Coordinate Systems

\( \mathbb{R}^3 \)

Plane where \( x = 0 \) is the \( y-z \) plane, etc.

Q: How do we graph \( x = y \) in \( \mathbb{R}^3 \)?
A: When in doubt, think in lower dim!

\( x = y \) in \( \mathbb{R}^2 \)

\( x = y \) in \( \mathbb{R}^3 \)

Right hand rule: curl fingers RH from \( x^+ \to y^+ \), thumb goes toward \( z^+ \).
Q: (Class) What are the solutions to

1. \( x^2 + y^2 + z^2 = 1 \)

A: \[ \begin{array}{c}
\text{sphere} \\
\text{radius} r = 1
\end{array} \]

2. \( x^2 + y^2 = 1 \)

A: \[ \begin{array}{c}
\text{cylinder} \\
r = 1
\end{array} \]

3. \( x^2 + y^2 + z^2 = 1, \quad x = y \)

A: The intersection of (1) with plane, so a \underline{circle} in \[ y = x \] plane

What is the point where \( z = 0 \) on the set? \( \left( \frac{1}{1}, \frac{1}{1}, 0 \right) \)

17. (Class) Find center and radius of sphere

\[
\begin{align*}
2x^2 + 2y^2 + 2z^2 &= 8x - 2y + 1 \\
x^2 + y^2 + z^2 &= 4x - 12y + \frac{1}{2} \\
x^2 - 4x + y^2 + z^2 + 12z + 36 &= \frac{1}{2} + 4 + 36 \\
(x - 2)^2 + y^2 + (z + 6)^2 &= 40.5 \\
\text{Center: } (2, 0, -6) \\
\text{Radius: } r = \sqrt{40.5}
\end{align*}
\]
§12.1 Distance formula in \( \mathbb{R}^n \), \( p_1 = (x_1, \ldots, x_n) \)
\( p_2 = (y_1, \ldots, y_n) \)

\[ \text{dist} = \| p_1 p_2 \| = \sqrt{n \sum (x_i - y_i)^2} \]

- This is just the Pythagorean theorem.
- Translate so \( p_1 = 0 \), \( p_2 = p_2 - p_1 \)

Class:
1. Find dist from \((111)\) to \((123)\).
   A: \( \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5} \)
2. Describe the region of example 7 in book, \( 1 \leq x^2 + y^2 + z^2 \leq 4 \).
3. What is the volume?
   A: a sphere of radius 2, centred at \( \vec{0} \), with a sphere of radius 1, "excised".
   So, volume = \( \frac{4}{3} \pi 2^3 - \frac{4}{3} \pi 1^3 \) .

§12.2 Vectors
A vector is a directed quantity
- has magnitude \( \vec{a} \) and direction
- e.g., speed (quantity)
- vs. velocity "+ directed."
Example

\[ \vec{b} \text{ has head at } (1, 2, 3) \]
\[ \text{tail at } (0,0,0) \]

Who is \( \vec{v} \)?

\[ \vec{a} + \vec{v} = \vec{b} \]

\[ (1,1,1) + \vec{v} = (1,2,3) \]

so \( \vec{v} = (0,1,2) \)

Point addition + subtraction of vectors is usual \underline{coordinate-wise} addition/subtraction.

Can rewrite (*) as \( \vec{v} = \vec{b} - \vec{a} \)

so put tail of \( \vec{v} \) at

head of \( \vec{a} \) to add them.

Remark:

\[ \vec{a} + \vec{v} = \vec{v} + \vec{a} \]

This agrees with what we know about coordinate-wise addition:

\[(a_1, b_1, \ldots, a_n, b_n) + (c_1, d_1, \ldots, c_n, d_n) \]
\[= (a_1 + c_1, b_1 + d_1, \ldots, a_n + c_n, b_n + d_n) \]

\[(b_1, a_1, \ldots, b_n, a_n) \]
\[= (b_1 + a_1, a_1 + b_1, \ldots, b_n + a_n) \]
\[(b_1, -b_n) + (c_1, -c_n) \]

Individual entries are called components.

Take home: vectors + and \(-\) act as expected.
12.2

- Multiplication by a scalar \( c \in \mathbb{R} \)
  
  \[ c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n) \quad \text{as expected.} \]

- Can encapsulate everything as "scalar mult and vector add, in work normally"
  
  e.g., \( (c_1 + c_2) \vec{v} = c_1 \vec{v} + c_2 \vec{v} \)

- Notation / Terminology
  
  - Unit vector is a vector with length one.
  - So unit vector in direction \( \vec{v} \) is just \( \vec{v} / \| \vec{v} \| \).

- Standard basis vectors
  
  - In \( \mathbb{R}^2 \) \( (10), (01) \)
  
  \[ \begin{array}{c}
  i \quad j \\
  e_1 \\ e_2
  \end{array} \]
  
  \( (a, b) = a \vec{e}_1 + b \vec{e}_2 \)

  - In \( \mathbb{R}^3 \) \( (100), (010), (001) \)

  \[ \begin{array}{c}
  i \\ j \\ k \\
  e_1 \\ e_2 \\ e_3
  \end{array} \]

  Notice: any vector is a combo of std basis vectors \( (a, b, c) = a \vec{e}_1 + b \vec{e}_2 + c \vec{e}_3 = a \vec{i} + b \vec{j} + c \vec{k} \).