Polar Coordinates \((x, y) = (r \cos \theta, r \sin \theta), \ 0 \leq r, \ 0 \leq \theta < 2\pi\)

Area

\[\text{area on RHS} = \frac{\Delta \theta}{2\pi} \left( \text{area of sector of radius } r + \Delta r \right) - \frac{\Delta \theta}{2\pi} \left( \text{area of sector of radius } r \right)\]

\[= \frac{\Delta \theta}{2\pi} \left[ \pi (r + \Delta r)^2 - \pi r^2 \right] = \frac{\Delta \theta}{2\pi} \left[ \pi \Delta r (2r + \Delta r) \right] = r \Delta r \Delta \theta \Delta \theta \Delta r, \Delta \theta \text{ very small}.\]

Thus in integrals

\[\text{dxdy} = r \text{drd} \theta.\]

Utility In interesting situations, have symmetry; want, or need, to take advantage of it.

Ex Volume above \(z = 0\), under \(z = 1 - x^2 - y^2\).

Recognize rotational symmetry:

\[x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2, \text{ or}\]

\[z = 1 - r^2.\]
Lies above \( \{ (x,y) \mid x^2 + y^2 \leq 1 \} = \\{ (r \cos \theta, r \sin \theta) \mid 0 < \theta < 2\pi \} \)

Volume \[
\begin{align*}
\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (1-r^2) \, r \, dr \, d\theta
\end{align*}
\]
\[
= 2\pi \int_{0}^{1} (1-r^2) \, dr
= 2\pi \left[ \frac{r^2}{2} - \frac{r^4}{4} \right]_{0}^{1}
= \frac{\pi}{2}.
\]

Ex: Electric charge is distributed over disk \( x^2 + y^2 \leq a \)
So charge density at \((x,y)\) is \( \sigma(x,y) = x + y + x^2 + y^2 \) Coulomb \( \frac{1}{4\pi} \).

Problem: Compute total charge on disk, i.e.
\[
\int \sigma(x,y) \, dA = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} (r \cos \theta + r \sin \theta + r^2) \, r \, dr \, d\theta.
\]
\[
= \left[ \text{Claim: sin, cos terms disappear - why?} \right] = 2\pi \int_{0}^{a} r^3 \, dr = \frac{8\pi a^4}{4} = 2\pi a^4.
\]
Gaussian "bell curve" \( f(x) = e^{-x^2} \).

It's a basic shape in probability, stat, econ, psych, whatever. [Because of Central Limit Theorem]

Need to know \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \). Otherwise can't normalize!

**Fact** There is no "reasonable" formula for \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \).

Provable by mathematical logic! But intuitively reasonable.

Gauss: Pass to seemingly "harder" problem.

Compute
\[
\left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \, dy \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} \, r \, dr \, d\theta
\]

\[
= 2\pi \int_{0}^{\infty} e^{-r^2} \, r \, dr = -\pi \int_{0}^{\infty} e^{-r^2} (2r) \, dr
\]

\[
= -\pi e^{-r^2} \bigg|_{0}^{\infty} = \pi. \; \text{So} \; \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.
\]

Then \( \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} \, dx = 1 \)!!
Center of Mass

Archimedes' Law of the Lever:

\[ d_1 M_1 = d_2 M_2. \]

Writing \( x \) for coord of center of mass, \( x_1, x_2 \) for pos of wts, get

\[ (x-x_1)M_1 = (x_2-x)M_2 \quad \text{or} \]

\[ x(M_1+M_2) = M_1x_1 + M_2x_2 \quad \text{or} \]

\[ x = \frac{M_1x_1 + M_2x_2}{M_1+M_2}, \]

Now add more wts, discover

\[ x = \frac{\sum m_i x_i}{\sum m_i} \quad \text{continuum limit} \]

\[ \int x \rho \, dA \quad \text{mass density} \]

In plane, center of mass of plate occupies, mass density, \( (\bar{x}, \bar{y}) = \frac{1}{\text{area}} \left( \int \int x \rho \, dA, \int \int y \rho \, dA \right) \)

\[ M_x = \int \int x \rho \, dA \quad \text{for moment of object around x axis} \]

\[ M_y = \int \int y \rho \, dA \]

\[ m = \int \int \rho \, dA \quad \text{mass}. \]
Ex. center of charge, before:

\[ M_X = \iint_{\text{disk}} y (x+\frac{y}{2}+x^2+y^2) \, dx \, dy \]

\[ = \int_0^{2\pi} \int_0^r (r^2 \cos \theta \sin \theta + \frac{r^3 \sin \theta}{2} + r^3 \cos ^2 \theta \sin \theta + r^3 \sin ^2 \theta) r \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^r (\frac{r^3 \sin \theta}{2} + r^3 \sin \theta + r^3 \sin \theta) r \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^r r^3 \sin ^2 \theta \, dr \, d\theta = 4 \int_0^{2\pi} \sin ^2 \theta \, d\theta \]

\[ = 4 \int_0^{\frac{2\pi}{2}} (\frac{1}{2} - \frac{1}{2} \cos 2\theta) \, d\theta = 4\pi. \]

By symmetry, \( M_Y = M_X! \)

Thus get center of mass charge

\[ \left( \frac{\frac{4\pi}{4\pi}}{\frac{8\pi}{8\pi}} \right) = \left( \frac{1}{2}, \frac{1}{2} \right). \]

Ex. C.O.M. of \( \bigotimes \) radius a, \( \rho = \rho r \) (polar).

Clearly \( \bar{X} = 0 \).

\[ m = \int_0^a \int_0^{\pi} r^2 \, dr \, d\theta = \pi \frac{a^3}{3}. \]

\[ \bar{M}_X = \int_0^a \int_0^{\pi} r^3 \sin \theta \cos \theta \, dr \, d\theta = \frac{a^4}{4} \int_0^{\pi} \sin \theta \cos \theta \, d\theta = \frac{a^4}{2}. \]

\[ \bar{Y} = \frac{M_x}{m} = \frac{a^4}{2/\pi \cdot a^3/3} = \frac{3a}{2\pi}. \]