Arrangements and Computations III: $\Lambda(V)$ and BGG

$$\begin{pmatrix}
\sum x_i & 0 & 0 & 0 \\
0 & \sum x_i & 0 & 0 \\
0 & 0 & \sum x_i & 0 \\
0 & 0 & 0 & \sum x_i \\
x_1+x_4+x_5 & 0 & 0 & 0 \\
0 & x_2+x_3+x_5 & 0 & 0 \\
0 & 0 & x_0+x_3+x_4 & 0 \\
0 & 0 & 0 & x_0+x_1+x_2 \\
-x_3 & 0 & 0 & x_3 \\
0 & x_4 & 0 & x_4 \\
0 & 0 & -x_5 & x_5 \\
x_0 & x_0 & 0 & 0 \\
x_2 & 0 & -x_2 & 0 \\
0 & x_1 & x_1 & 0
\end{pmatrix}.$$

$$4 \quad 10 \quad 15 \quad 20 \quad 25 \quad \cdots$$

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Let A be the Orlik-Solomon algebra of $\mathbb{C}^\ell \setminus \mathcal{A}$, with $|\mathcal{A}| = n$. For each $a = \sum a_i e_i \in A_1$, we consider the complex (A, a).

The i^{th} term is A_i, and differential is $\wedge a$:

$$(A,a)\colon\ 0 \longrightarrow A_0 \xrightarrow{a} A_1 \xrightarrow{a} A_2 \xrightarrow{a} \cdots \xrightarrow{a} A_\ell \longrightarrow 0\,.$$

Arose in

- hypergeometric functions (**Aomoto**)
- cohomology with local system coefficients

–**Esnault, Schechtman, Viehweg**

–**Schechtman, Terao, Varchenko**

The *resonance varieties* of $\mathcal{A}$ are the loci of points $a = \sum_{i=1}^n a_i e_i \leftrightarrow (a_1 : \cdots : a_n) \in \mathbb{P}^{n-1}$ for which (A, a) fails to be exact, that is:.

Definition 1 *For each $k \geq 1$,*

$$R^k(\mathcal{A}) = \{a \in \mathbb{P}^{n-1} \mid H^k(A, a) \neq 0\}.$$

Yuzvinsky: for generic a, (A, a) is exact.

Definition 2 Π *partition of $\mathcal{A}$ is neighborly if $\forall Y \in L_2(\mathcal{A})$, π block of Π,*

$$\mu(Y) \leq |Y \cap \pi| \longrightarrow Y \subseteq \pi.$$

Falk: proved that components of $R^1(\mathcal{A})$ arise from neighborly partitions, and conjectured that $R^1(\mathcal{A})$ is a union of linear components.

This was proved by
- **Cohen–Suciu** and by
- **Libgober–Yuzvinsky** $R^1(\mathcal{A}) = \amalg L_i^+$
- **Cohen–Orlik** also true for $R^{\geq 2}(\mathcal{A})$
- **Falk** can fail if characteristic $\neq 0$.

Libgober–Yuzvinsky connects $R^1(\mathcal{A})$ to pencils/nets/webs; recent work in this area by:
- **Falk–Yuzvinsky**
- **Pereira–Yuzvinsky**

Recall conjectural connection to LCS ranks ϕ_k:

Conjecture 3 (Suciu) *Under certain conditions,*

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k} = \prod_{L_i \in R^1(\mathcal{A})} (1 - (dim(L_i)t)$$

Example 4 Let $\mathcal{A} = V(xy(x-y)z) \subseteq \mathbb{P}^2$, and $E = \Lambda(\mathbb{C}^4)$, with generators $e_1, \dots, e_4$. The Orlik-Solomon algebra

$$A = E/\langle \partial(e_1e_2e_3), \partial(e_1e_2e_3e_4)\rangle, \text{ with}$$

$$\partial(e_1e_2e_3) \;=\; e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3$$

To compute $R^1(\mathcal{A})$, we need only the first two differentials in the Aomoto complex. Use $e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$ as a basis for A_2.

$$e_1 \mapsto e_1 \wedge (\sum_{i=1}^{4} a_i e_i) = a_2e_{12} + a_3e_{13} + a_4e_{14}.$$

Since $e_{12} = e_{13} - e_{23}$, $a_2e_{12} = a_2(e_{13} - e_{23})$,

giving $(a_2 + a_3)e_{13} + a_4e_{14} - a_2e_{23}$. compute!

$$0 \longrightarrow \mathbb{C}^1 \xrightarrow{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}} \mathbb{C}^4 \xrightarrow{\begin{bmatrix} a_2+a_3 & -a_1 & -a_1 & 0 \\ a_4 & 0 & 0 & -a_1 \\ -a_2 & a_1+a_3 & -a_2 & 0 \\ 0 & a_4 & 0 & -a_2 \\ 0 & 0 & a_4 & -a_3 \end{bmatrix}} \mathbb{C}^5$$

Letting $a = \sum_{i=1}^{n} a_i e_i$, we have

$$\begin{array}{rcc} R^1(\mathcal{A}) & \leftrightarrow & H^1(A, \wedge a) \\ & \leftrightarrow & \exists b \in E_1 \mid a \wedge b \text{ vanishes in } A_2 \\ & \leftrightarrow & \exists b \in E_1 \mid a \wedge b \in I_2 \\ & \leftrightarrow & \text{decomposable 2-tensors in } I_2 \\ & \leftrightarrow & \mathbb{P}(I_2) \cap \mathrm{Gr}(2, E_1) \subseteq \mathbb{P}(\bigwedge^2 E_1) \end{array}$$

I_2 is determined by the intersection lattice $L(\mathcal{A})$ in rank ≤ 2, so to study $R^1(\mathcal{A})$, let $\mathcal{A} \subseteq \mathbb{P}^2$. Grassmannian gives fastest compution of $R^1(\mathcal{A})$.

Problem Code up for $R^{\geq 2}(\mathcal{A})$ (Segre map).

Note interesting connection to syzygies. Since

$a \wedge b \in I_2 \longrightarrow a \wedge b = \sum c_i f_i,\ c_i \in \mathbb{C}, f_i \in I_2,$ the relations $a \wedge a \wedge b = 0 = b \wedge a \wedge b$ yield linear syzygies on I_2:

$$\sum a c_i f_i = 0 = \sum b c_i f_i.$$

That is,

$$R^1(\mathcal{A}) \text{ is related to } Tor_2^E(A, \mathbb{C})_3$$

Example 5 For $\mathcal{A} = V(xy(x-y)z) \subseteq \mathbb{P}^2$, the Orlik-Solomon algebra is just

$$A = E/\partial(e_1e_2e_3),$$

since the relation $\partial(e_1e_2e_3e_4)$ is redundant:

$$\partial(e_1e_2e_3e_4) = e_1 \wedge \partial(e_1e_2e_3) - e_4\partial(e_1e_2e_3)$$

Observe that

$$e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3 = (e_1 - e_2) \wedge (e_2 - e_3)$$

This means that the line

$$s(e_1 - e_2) + t(e_2 - e_3) \subseteq R^1(\mathcal{A}) \subseteq \mathbb{P}(E_1)$$

Parametrically, this may be written

$$(s : t - s : -t : 0) = V(a_4, a_1 + a_2 + a_3)$$

Such components of $R^1(\mathcal{A})$ are called local. (Compute) the corresponding linear syzygies.

WHO CARES? Conjecturally, $R^1(\mathcal{A})$ is (sometimes) connected to the LCS ranks. But it is always connected to the Chen ranks! Introduced by K.T. Chen, these are the LCS ranks of the maximal metabelian quotient of G:

$$\theta_k(G) := \phi_k(G/G''),$$

where $G' = [G, G]$.

Conjecture 6 (Suciu) *Let $G = G(\mathcal{A})$ be an arrangement group, and let h_r be the number of components of $R^1(\mathcal{A})$ of dimension r. Then, for $k \gg 0$:*

$$\theta_k(G) = (k-1)\sum_{r\geq 1} h_r \binom{r+k-1}{k}.$$

For Example 3, $R^1(\mathcal{A}) \simeq \mathbb{P}^1$ and thus

$$\theta_k(G) = (k-1).$$

How to determine the Chen ranks? The Alexander invariant G'/G'' is a module over $\mathbb{Z}[G/G']$. For arrangements, $\mathbb{Z}[G/G']$ = Laurent polynomials in n-variables.

$$\text{Massey: } \sum_{k\geq 0} \theta_{k+2}\, t^k = HS(\text{gr } G'/G'' \otimes \mathbb{Q}, t)$$

Easier to work with is the linearized Alexander invariant B of **Cohen-Suciu**

$$(A_2 \oplus E_3) \otimes S \xrightarrow{\Delta} E_2 \otimes S \longrightarrow B \longrightarrow 0\,, \text{ where}$$

Δ is built from Koszul diff. and $(E_2 \to A_2)^t$.

Theorem 7 (Cohen-Suciu)

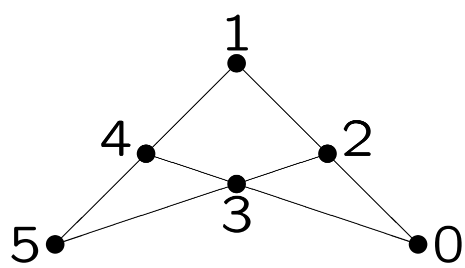
$$V(\mathit{ann}\ B) = R^1(\mathcal{A})$$

Theorem 8 (Papadima-Suciu) *For $k \geq 2$,*

$$\sum_{k\geq 2} \theta_k\, t^k = HS(B, t).$$

In particular, the Chen ranks are combinatorially determined, and depend only on $L(\mathcal{A})$ in rank ≤ 2.

Example 9 Recall the matroid for A_3 is:



For A_3, B is the cokernel of the matrix on the first slide. (compute) $R^1(A_3) =$

$$V(x_1 + x_4 + x_5, x_0, x_2, x_3) \amalg$$
$$V(x_2 + x_3 + x_5, x_0, x_1, x_4) \amalg$$
$$V(x_0 + x_3 + x_4, x_1, x_2, x_4) \amalg$$
$$V(x_0 + x_1 + x_2, x_3, x_4, x_5) \amalg$$
$$V(x_0 + x_1 + x_2, x_0 - x_5, x_1 - x_3, x_2 - x_4).$$

and (compute) the Hilbert Series of B:

$$(4t^2+2t^3-t^4)/(1-t)^2 = 4t^2+10t^3+15t^4+20t^5+\cdots$$

Magic Trick! (compute) $Tor_i^E(A_3, \mathbb{C})_{i+1}$

Magic Trick! (compute) free resolution of the cokernel of last map in the Aomoto complex.

Theorem 10 (Eisenbud-Popescu-Yuzvinsky) *For an arrangement $\mathcal{A}$, the Aomoto complex is exact, as a sequence of S-modules:*

$$0 \longrightarrow A_0 \otimes S \xrightarrow{\cdot a} A_1 \otimes S \xrightarrow{\cdot a} \cdots \xrightarrow{\cdot a} A_\ell \otimes S \longrightarrow F(A) \longrightarrow 0\,.$$

Theorem 11 (–, Suciu) *The linearized Alexander invariant B is functorially determined by the Orlik-Solomon algebra:*

$$B \cong \mathrm{Ext}_S^{\ell-1}(F(A), S).$$

Use this, localization, and the result of Libgober-Yuzvinsky that $R^1(\mathcal{A}) = \amalg L_i$ to obtain:

Theorem 12 (–, Suciu) *For $k \gg 0$,*

$$\theta_k(G) \geq (k-1) \sum_{L_i \in R^1(\mathcal{A})} \binom{\dim L_i + k - 1}{k}.$$

Problem Prove the remaining inequality! Note: $\theta_k(G)$ is polynomial in k, of degree $= \dim R^1(\mathcal{A})$.

WHAT MAKES ALL THIS WORK IS BGG: the Bernstein-Gelfand-Gelfand correspondence.

Let $S = Sym(V^*)$ and $E = \bigwedge(V)$. BGG is an isomorphism between derived categories of

- bounded cpxs of coherent sheaves on $\mathbb{P}(V^*)$.
- bounded cpxs of f.gen'd, graded E–modules.

From this, can extract functors
$\mathbf{R}$: f.gen'd, graded S-modules $\longrightarrow$ linear free E-complexes.

$\mathbf{L}$: f.gen'd, graded E-modules $\longrightarrow$ linear free S-complexes.

Point: can translate problems to possibly simpler setting. For example, we'll see this gives a fast way to compute sheaf cohomology, using Tate resolutions.

P a f'gend, graded E-module, then $\mathbf{L}(P)$ is the complex

$$\cdots \longrightarrow S \otimes P_{i+1} \xrightarrow{\cdot a} S \otimes P_i \xrightarrow{\cdot a} S \otimes P_{i-1} \xrightarrow{\cdot a} \cdots,$$

where $a = \sum_{i=1}^{n} x_i \otimes e_i$, so that $1 \otimes p \mapsto \sum x_i \otimes e_i \wedge p$

Note: elts of V^* deg $= 1$, elts of V deg $= -1$.

Example 13 $P = E = \bigwedge \mathbb{C}^3$. Then we have

$$0 \longrightarrow S \otimes E_0 \longrightarrow S \otimes E_1 \longrightarrow S \otimes E_2 \longrightarrow S \otimes E_3 \longrightarrow 0\,.$$

Clearly $1 \mapsto \sum_1^3 x_i \otimes e_i$. For d_1

$$e_1 \mapsto -x_2 e_{12} - x_3 e_{13}$$

$$e_2 \mapsto x_1 e_{12} - x_3 e_{23}$$

$$e_3 \mapsto x_1 e_{13} + x_2 e_{23}$$

d_2 : $e_{12} \mapsto x_3 e_{123}$, $e_{13} \mapsto -x_2 e_{123}$ $e_{23} \mapsto x_1 e_{123}$

Thus, $\mathbf{L}(E)$ is

$$S^1 \xrightarrow{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} S^3 \xrightarrow{\begin{bmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{bmatrix}} S^3 \xrightarrow{\begin{bmatrix} x_3 & -x_2 & x_1 \end{bmatrix}} S^1$$

The Koszul complex!

M a f'gend, graded S-module, then $\mathbf{R}(M)$ is the complex

$$\cdots \longrightarrow \hat{E} \otimes M_{i-1} \xrightarrow{\cdot a} \hat{E} \otimes M_i \xrightarrow{\cdot a} \hat{E} \otimes M_{i+1} \xrightarrow{\cdot a} \cdots,$$

where $a = \sum_{i=1}^{n} e_i \otimes x_i$, so $1 \otimes m \mapsto \sum e_i \otimes x_i \cdot m$, and $\hat{E}$ is the $\mathbb{C}$-dual of E:

$$\hat{E} \simeq E(n) = Hom_{\mathbb{C}}(E, \mathbb{C}).$$

Just as $\mathbf{L}(P) = S \otimes_{\mathbb{C}} P$, $\mathbf{R}(M) = Hom_{\mathbb{C}}(E, M)$.

Example 14 $M = \mathbb{C}[x_0, x_1]/\langle x_0 x_1, x_0^2 \rangle$. Then

$$0 \longrightarrow E \otimes M_0 \longrightarrow E \otimes M_1 \longrightarrow E \otimes M_2 \longrightarrow E \otimes M_3 \longrightarrow \cdots$$

$$1 \mapsto e_0 \otimes x_0 + e_1 \otimes x_1$$

$$x_0 \mapsto e_0 \otimes x_0^2 + e_1 \otimes x_0 x_1$$

$$x_1 \mapsto e_0 \otimes x_0 x_1 + e_1 \otimes x_1^2$$

$$x_1^n \mapsto e_0 \otimes x_0 x_1^n + e_1 \otimes x_1^{n+1}$$

Thus, $\mathbf{R}(M)$ is

$$E(2)^1 \xrightarrow{\begin{bmatrix} e_0 \\ e_1 \end{bmatrix}} E(3)^2 \xrightarrow{\begin{bmatrix} 0 & e_1 \end{bmatrix}} E(4)^1 \xrightarrow{\begin{bmatrix} e_1 \end{bmatrix}} E(5)^1 \xrightarrow{\begin{bmatrix} e_1 \end{bmatrix}} \cdots$$

This complex is exact, except at the second step. Obviously the kernel of

$$\begin{bmatrix} 0 & e_1 \end{bmatrix}$$

is generated by $\alpha = [1, 0]$ and $\beta = [0, e_1]$, with relations $im(d_1) = \beta + e_0\alpha = 0, e_1\beta = 0$, so that

$$H^1(\mathbf{R}(M)) \simeq E(3)/e_0 \wedge e_1$$

Compute this, and compute the free resolution of M. This illustrates

Theorem 15 (Eisenbud-Fløystad-Schreyer)

$$H^j(\mathbf{R}(M))_{i+j} = Tor_i^S(M, \mathbb{C})_{i+j}.$$

Corollary 16 *The Castelnuovo-Mumford regularity of M is $\leq d$ iff $H^i(\mathbf{R}(M)) = 0$ for all $i > d$.*

What can be said about higher resonance varieties? **Cohen–Orlik** proved that for $k \geq 2$,

$$R^k(\mathcal{A}) = \bigcup L_i \text{ linear.}$$

Suciu showed union need not be disjoint.

Theorem 17 (Eisenbud-Popescu-Yuzvinsky) *Resonance persists:* $p \in R^k(\mathcal{A}) \longrightarrow p \in R^{k+1}(\mathcal{A})$.

The key observation is $a \in R^k(\mathcal{A}) \subseteq \mathbb{P}(E)$ means

$$H^k(A, a) \neq 0 \leftrightarrow Tor^S_{\ell-k}(F(A), S/I(p)) \neq 0.$$

The result follows from interpreting this in terms of Koszul cohomology.

Theorem 18 (Denham, –) *As for* $R^1(\mathcal{A})$*, higher resonance may be interpreted via* Ext*:*

$$R^k(\mathcal{A}) = \bigcup_{k' \leq k} V(\operatorname{ann} \operatorname{Ext}^{\ell-k'}(F(A), S)).$$

Differentials in free resolution can be analyzed using BGG and Grothendieck spectral sequence (work in progress, Denham, –).

For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^d$, there is a f'gend, graded S-module M whose sheafification is $\mathcal{F}$. If $\mathcal{F}$ has Castelnuovo-Mumford regularity r, then the **Tate resolution** of $\mathcal{F}$ is obtained by splicing the complex $\mathbf{R}(M_{\geq r})$:

$$0 \longrightarrow \widehat{E} \otimes M_r \xrightarrow{d^r} \widehat{E} \otimes M_{r+1} \longrightarrow \widehat{E} \otimes M_{r+2} \longrightarrow \cdots,$$

with a free resolution $P_\bullet$ for the kernel of d^r:

$$\begin{array}{ccccccccc} \cdots \longrightarrow P_1 \longrightarrow P_0 & & \longrightarrow & & \widehat{E} \otimes M_r \longrightarrow \widehat{E} \otimes M_{r+1} \longrightarrow \cdots \\ & \searrow & & \nearrow & \\ & & \ker(d^r) & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array}$$

By Corollary 16, $\mathbf{R}(M_{\geq r})$ is exact except at the first step, so this yields an exact complex of free E-modules.

Example 19 Since $M = S$ has regularity zero, we obtain Cartan resolutions in both directions, with splice map $E \to \widehat{E} = E(d+1)$ multiplication by $e_0 \wedge e_1 \wedge \cdots \wedge e_d = \ker \left[e_0, \ \cdots \ , e_d \right]^t$.

Theorem 20 (Eisenbud-Fløystad-Schreyer) *The i^{th} free module T^i in a Tate resolution for $\mathcal{F}$ satisfies*

$$T^i = \bigoplus_j \widehat{E} \otimes H^j(\mathcal{F}(i-j)).$$

Example 21 Twisted cubic $I \subseteq S = \mathbb{C}[x, y, z, w]$

$$0 \longrightarrow S(-3)^2 \xrightarrow{\begin{bmatrix} -z & w \\ y & -z \\ -x & y \end{bmatrix}} S(-2)^3 \xrightarrow{\begin{bmatrix} y^2-xz & yz-xw & z^2-yw \end{bmatrix}} S \longrightarrow S/I$$

Display as a *betti table*:

$$b_{ij} = \dim_{\mathbb{C}} \mathrm{Tor}_i^S(M, \mathbb{C})_{i+j}.$$

total	1	3	2
0	1	–	–
1	–	3	2

This has regularity one, so now we can (compute) the Tate resolution:

Plugging these numbers into Theorem 20, we see that

i	-3	-2	-1	0	1	2
$h^1(\mathcal{F}(i))$	8	5	2	0	0	0
$h^0(\mathcal{F}(i))$	0	0	0	1	4	7

Does this make sense?

$$\mathcal{F} = \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}(3)$$

so

$$h^1(\mathcal{F}(i)) = h^1(\mathcal{O}_{\mathbb{P}^1}(3i)) = h^0(\mathcal{O}_{\mathbb{P}^1}(-3i-2))$$

and

$$h^0(\mathcal{F}(i)) = h^0(\mathcal{O}_{\mathbb{P}^1}(3i)) = 3i+1,\ i \geq 0$$

THE END! THANK YOU!

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