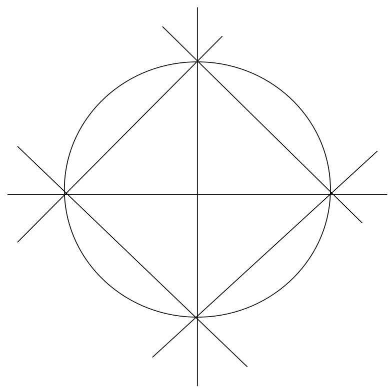
Arrangements and Computations I: $Sym(V^*)$



$(1, 2, 3)$ and $(1, 2, 5)$

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August 19, 2009

§Basics

$$\text{Let } \mathcal{A} \subseteq V = \mathbb{C}^\ell$$

be a central arrangement with $|\mathcal{A}| = n$, and $S = Sym(V^*)$.

$$S = \bigoplus_{i \in \mathbb{Z}} S_i$$

is a $\mathbb{Z}$-graded ring:

$$s_i \in S_i \text{ and } s_j \in S_j \longrightarrow s_i \cdot s_j \in S_{i+j}$$

Similar definition for a graded S-module M. $S_0 = \mathbb{C}$, so M_i is a $\mathbb{C}$–vector space.

Definition 1 *The Hilbert Function*

$$HF(M, i) = \dim_{\mathbb{C}} M_i.$$

Definition 2 *The Hilbert Series*

$$HS(M, i) = \sum_{\mathbb{Z}} \dim_{\mathbb{C}} M_i t^i.$$

Notation: $M(i)_j = M_{i+j}$.

Exercise: $HS(\mathbb{C}[x_1, \ldots, x_\ell], t) = \frac{1}{(1-t)^\ell}$.

Example 3 $S = \mathbb{C}[x, y]$, $M = S/\langle x^2, xy\rangle$. Then

i	M_i	$M(-2)_i$
0	1	0
1	x, y	0
2	y^2	1
3	y^3	x, y
4	y^4	y^2

$$HS(M, i) = \frac{1 - 2t^2 + t^3}{(1-t)^2}$$

$$HS(M(-2), i) = \frac{t^2(1 - 2t^2 + t^3)}{(1-t)^2}$$

Makes sense: $S(-i)$ has generator in degree i.

Compute from *free resolution*:

$$0 \longrightarrow S(-3) \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} S(-2)^2 \xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} S \longrightarrow S/I$$

$$\begin{array}{l} e_1 \mapsto x^2 \\ e_2 \mapsto xy \end{array}$$

$$HS(M,i) = \frac{t^3 - 2t^2 + 1}{(1-t)^2}$$

Example 4 Twisted cubic $I \subseteq S = \mathbb{C}[x,y,z,w]$

$$0 \longrightarrow S(-3)^2 \xrightarrow{\begin{bmatrix} -z & w \\ y & -z \\ -x & y \end{bmatrix}} S(-2)^3 \xrightarrow{\begin{bmatrix} y^2-xz & yz-xw & z^2-yw \end{bmatrix}} S \longrightarrow S/I$$

Display as a *betti table*:

$$b_{ij} = \dim_{\mathbb{C}} \operatorname{Tor}_i^S(M,\mathbb{C})_{i+j}.$$

total	1	3	2
0	1	–	–
1	–	3	2

$$b_{21} = \dim_{\mathbb{C}} \operatorname{Tor}_2^S(S/I,\mathbb{C})_3 = 2.$$

§D($\mathcal{A}$) and freeness

For each i, fix $V(l_i) = H_i \in \mathcal{A}$. Let $Q_{\mathcal{A}} = \prod_{i=1}^{n} l_i$

Definition 5 $D(\mathcal{A}) = \{\theta \in Der_C(S) | \theta(l_i) \in \langle l_i \rangle\}$ *$\forall\ l_i$ with $V(l_i) \in \mathcal{A}$. $\mathcal{A}$ is free $\leftrightarrow$ $D(\mathcal{A})$ is free.*

Exercise: if $\theta_E = \sum_{i=1}^{\ell} x_i \partial/\partial x_i$, then

$$D(\mathcal{A}) \simeq S \cdot \theta_E \oplus syz(Jac(Q_{\mathcal{A}})),$$

where syz is the syzygy module and $Jac(Q_{\mathcal{A}})$ is the jacobian ideal of $Q_{\mathcal{A}}$.

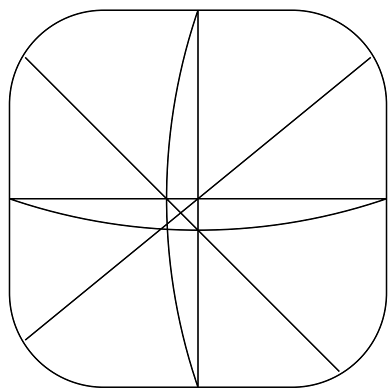
Proposition 6 (K. Saito) *$\mathcal{A}$ is free exactly when there exist ℓ elements*

$$\theta_i = \sum_{j=1}^{\ell} f_{ij} \frac{\partial}{\partial x_j} \in D(\mathcal{A})$$

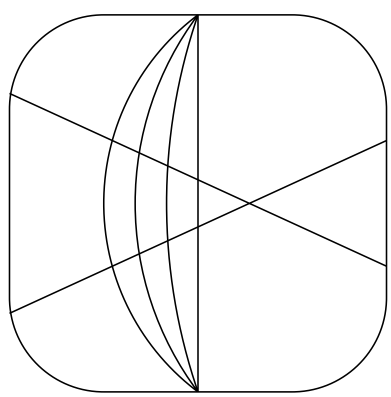
such that the determinant of the matrix $[f_{ij}]$ is a nonzero constant multiple of the defining polynomial $Q_{\mathcal{A}}$.

Compute $D(\mathcal{A})$ for arrangements in $\mathbb{P}^2$:

Example 7 [A3 and Nonfano]



Example 8 [S3]



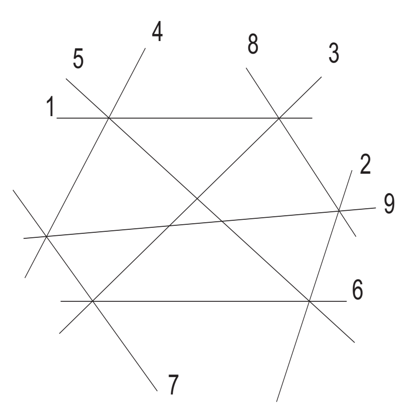
$$\pi(D_3, t) = (1+t)(1+3t)^2 = \pi(S_3, t).$$

Theorem 9 (Terao) *If* $D(\mathcal{A}) \simeq \bigoplus_{i=1}^{\ell} S(-a_i)$, *then*

$$\pi(\mathcal{A}, t) = \prod(1 + a_i t) = \sum \dim_{\mathbb{C}} H^i(\mathbb{C}^\ell \setminus \mathcal{A}) t^i.$$

Conjecture 10 (Terao) *If* $char = 0$, *then freeness of* $D(\mathcal{A})$ *depends only on* $L_{\mathcal{A}}$.

Example 11 [ZieglerAB] Compute $D(\mathcal{A})$ for arrangement



where 6 triple points lie on/off a conic.

Definition 12 *$D^p(\mathcal{A}) \subseteq \Lambda^p(Der_{\mathbb{C}}(S))$ consists of θ such that*

$$\theta(l_i, f_2, \ldots, f_p) \in \langle l_i \rangle, \forall\ V(l_i) \in \mathcal{A}, f_i \in S.$$

Theorem 13 (Solomon-Terao) $\chi(\mathcal{A}, t) =$

$$(-1)^{\ell} lim_{x \to 1} \sum_{p \geq 0} HS(D^p(\mathcal{A}); x)(t(x-1)-1)^p.$$

Problem How does

$$\operatorname{pdim} D^p(\mathcal{A})$$

depend on $L_{\mathcal{A}}$?

Theorem 14 (Yuzvinsky) *If $\hat{\mathcal{A}}$ a closed subarrangement of $\mathcal{A}$, then* $\operatorname{pdim} D(\mathcal{A}) \geq \operatorname{pdim} D(\hat{\mathcal{A}})$.

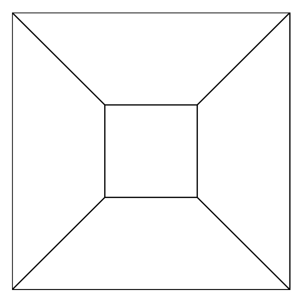
Aside from this, virtually nothing is known!

G a (simple) graph on ℓ vertices and edges E.
Put $\mathcal{A}_G = \{z_i - z_j = 0 \mid (i,j) \in \mathsf{E} \subseteq \mathbb{C}^\ell\}$

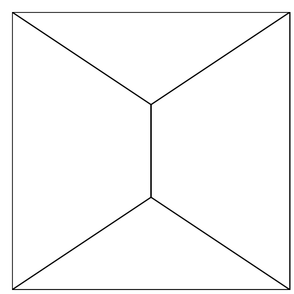
Stanley $\mathcal{A}_G$ is supersolvable $\leftrightarrow$ G is chordal.

Kung,– Induced k-cycle $\rightarrow$ pdim $D(\mathcal{A}_G) \geq k-3$

Example 15 G has induced 6-cycle (compute)



Example 16 G has induced 4-cycle (compute)



Problem Graph theory formula for pdim $D(\mathcal{A}_G)$?

Proving freeness: three ways

1. Addition-Deletion Theorem **(Terao)**
$(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ a triple: $\mathcal{A}' = \mathcal{A} \setminus H, \mathcal{A}'' = \mathcal{A}|_H$.
Any two below imply third.

- $D(\mathcal{A}) \simeq \oplus_{i=1}^{n} S(-b_i)$
- $D(\mathcal{A}') \simeq S(-b_n + 1) \oplus_{i=1}^{n-1} S(-b_i)$
- $D(\mathcal{A}'') \simeq \oplus_{i=1}^{n-1} S/L(-b_i)$

2. Supersolvable **(Terao, via AD)**

3. Multiarrangements **(Yoshinaga)**
$\mathcal{A} \subseteq \mathbb{P}^2$ is free $\leftrightarrow$
- $\pi(\mathcal{A}, t) = (1 + t)(1 + at)(1 + bt)$ and
- $D(\mathcal{A}|_H, \mathbf{m}) \simeq S/L(-a) \oplus S/L(-b)$,

holds $\forall H = V(L) \in \mathcal{A}$, with $\mathbf{m}(H_i) = \mu_{\mathcal{A}}(H \cap H_i)$.

§**Multiarrangements**

Definition 17 $(\mathcal{A}, \mathbf{m})$*: assign a multiplicity* m_i *to each hyperplane.*

$$D(\mathcal{A}, \mathbf{m}) = \{\theta \mid \theta(l_i) \in \langle l_i^{m_i} \rangle\}.$$

Example 18 [Ziegler, again!] Consider the two multiarrangements in $\mathbb{P}^1$

$\mathcal{A}_1 = (1,0),(0,1),(1,1),(1,-1)) \ast$ in $\mathbb{A}^2$

$\mathcal{A}_2 = (1,0),(0,1),(1,1),(1,a))$ $(a \neq -1)$

To compute $D(\mathcal{A}_1,(1,1,3,3))$, we must find all $\theta = f_1(x,y)\partial/\partial x + f_2\partial/\partial y$ such that

$$\theta(x) \in \langle x\rangle,\ \theta(x+y) \in \langle x+y\rangle^3$$
$$\theta(y) \in \langle y\rangle,\ \theta(x-y) \in \langle x-y\rangle^3$$

So compute kernel of

$$\begin{bmatrix} 1 & 0 & x & 0 & 0 & 0 \\ 0 & 1 & 0 & y & 0 & 0 \\ 1 & 1 & 0 & 0 & (x+y)^3 & 0 \\ 1 & -1 & 0 & 0 & 0 & (x-y)^3 \end{bmatrix}$$

Theorem 19 (Abe, Terao, Wakefield)

$$\Psi(\mathcal{A}, \mathbf{m}, t, q) = \sum_{p=0}^{\ell} HS(D^p(\mathcal{A}, \mathbf{m}, q))(t(q-1)-1)^p$$

$$\chi((\mathcal{A}, \mathbf{m}), t) = (-1)^{\ell}\Psi(\mathcal{A}, \mathbf{m}, t, 1).$$

If $D^1(\mathcal{A}, \mathbf{m}) \simeq \oplus S(-d_i)$ *then*

$$\chi((\mathcal{A}, \mathbf{m}), t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

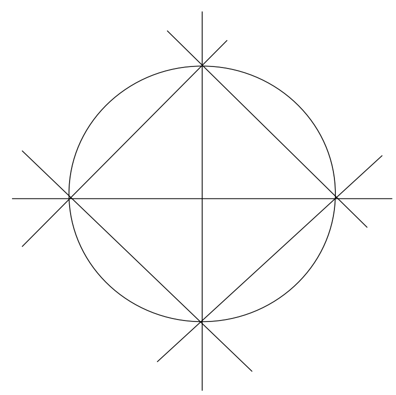
Abe, Terao, Wakefield also prove an addition-deletion theorem for multiarrangements, using *Euler multiplicity* for the restriction.

Hilbert-Burch Thm $\longrightarrow$ any $(\mathcal{A}, \mathbf{m}) \subseteq \mathbb{P}^1$ is free.
Problem $\exists$ other arrangements which are free for any $\mathbf{m}$? No! **Abe, Terao, Yoshinaga**: any such is a product of 1 and 2-dim arrangements.
Problem Characterize pdim $D(\mathcal{A}, \mathbf{m})$.
Problem Supersolvability for multiarrangements?

§Arrangements of hypersurfaces

Saito's criterion still holds. Are there other freeness theorems? Addition-Deletion theorem (even for $\mathcal{C} \subseteq \mathbb{P}^2$)?

Example 20 For the arrangement $\mathcal{C} \subseteq \mathbb{P}^2$



Compute $D(\mathcal{C})$

For a good theory, must control singularities.

Definition 21 *Plane curve singularity is quasi-homogeneous $\leftrightarrow$ $\exists$ holo Δ vars so $f(x,y) = \sum c_{ij}x^iy^j$ is weighted homogeneous: $\exists$ $\alpha, \beta \in \mathbb{Q}$ s.t. $\sum c_{ij}x^{i\cdot\alpha}y^{j\cdot\beta}$ is homogeneous.*

Definition 22 *The Milnor number at* $(0,0)$ *is*

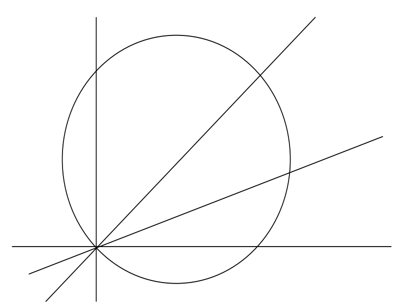
$$\mu_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle.$$

The Tjurina number at $(0,0)$ *is*

$$\tau_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \rangle.$$

for general p, just translate. For $V(Q) \subseteq \mathbb{P}^2$, note the degree of $Jac(Q) = \sum_{p \in sing(V(Q))} \tau_p$.

Example 23 Let $\mathcal{C}$ be as below:

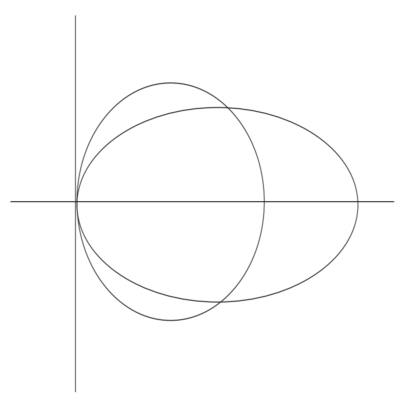


If p an ordinary sing with k distinct branches, then $\mu_p(C) = (k-1)^2$, so the sum of the Milnor numbers is 20. Compute deg(J). What happens at the origin?

Theorem 24 (K. Saito) *If $C = V(f)$ has an isolated sing. at the origin, then*

$$f \in Jac(f) \leftrightarrow f \text{ is quasihomogeneous.}$$

For a qhomogeneous line/conic arrangement, $\exists$ addition/deletion theorem (–,Tohaneanu). Compute $D(\mathcal{C})$ for



Can use AD to show this. Now change $\mathcal{C}$ to $\mathcal{C}'$ via: $y = 0 \longrightarrow x - 13y = 0$ and compute $D(\mathcal{C}')$.

§Orlik–Terao algebra

The Orlik–Terao algebra is (almost) a symmetric version of the Orlik-Solomon algebra. If codim $\cap_{j=1}^{m} H_{i_j} < m$, then $\exists$ c_{i_j} with

$$\sum_{j=1}^{m} c_{i_j} \cdot l_{i_j} = 0 \text{ a dependency.}$$

$$I_{\mathcal{A}} = \langle \sum_{j=1}^{m} c_{i_j}(y_{i_1} \cdots \widehat{y}_{i_j} \cdots y_{i_m}) \mid \text{ over all deps}\rangle$$

Definition 25 *The Orlik-Terao algebra is*

$$C(\mathcal{A}) = \mathbb{C}[x_1, \ldots, x_n]/I_{\mathcal{A}}.$$

Example 26 $\mathcal{A} = V(x_1 \cdot x_2 \cdot x_3 \cdot (x_1 + x_2 + x_3))$, the only dependency is

$$l_1 + l_2 + l_3 - l_4 = 0, \text{ thus } C(\mathcal{A}) =$$

$$\mathbb{C}[y_1, y_2, y_3, y_4]/\langle y_2y_3y_4 + y_1y_3y_4 + y_1y_2y_4 - y_1y_2y_3\rangle.$$

Artinian version of Orlik-Terao algebra is

$$AOT = C(\mathcal{A})/\langle x_1^2, \ldots, x_n^2 \rangle.$$

Theorem 27 (Orlik-Terao)

$$HS(AOT) = \pi(\mathcal{A}, t)$$

answering a question of Aomoto. For the previous example, Hilbert series of AOT is

$$1 + 4t + \binom{4}{2}t^2 + (\binom{4}{3} - 1)t^3$$

Theorem 28 (Terao)

$$HS(OT, t) = \pi\Big(\mathcal{A}, \frac{t}{1-t}\Big).$$

Can show that

$$0 \to I_{\mathcal{A}} \to \mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\phi} \mathbb{C}\left[\frac{1}{l_1}, \ldots, \frac{1}{l_n}\right] \to 0,$$

so $V(I_{\mathcal{A}}) \subseteq \mathbb{P}^{n-1}$ is irreducible and rational. **Problem** What is the geometry of $V(I_{\mathcal{A}})$?

Definition 29 *$\mathcal{A}$ is 2-formal if all dependencies are generated by dependencies among three hyperplanes.*

Theorem 30 (Falk-Randell) *$K(\pi, 1)$ and qOS arrangements are 2-formal.*

Theorem 31 (Yuzvinsky) *Free arrangements are 2-formal.*

WARNING! ZieglerA is 2-formal, ZieglerB is not. How to detect?

Formality involves the actual dependencies, which are captured by $C(\mathcal{A})$! Compute OT and AOT for Ziegler arrangements.

Theorem 32 (–,Tohaneanu)

$$\mathcal{A} \textit{ is 2-formal } \leftrightarrow \operatorname{codim}(I_2) = n - \ell.$$

What about other information? Is $V(I_{\mathcal{A}})$ smooth? Compute for $V(y_2y_3y_4+y_1y_3y_4+y_1y_2y_4-y_1y_2y_3)$.

Notice that the map $\phi(y_i) = \frac{1}{l_i}$ can be rewritten as

$$y_i \mapsto \alpha_i = l_1 \cdot l_2 \cdots \widehat{l_i} \cdots l_n.$$

For simplicity, restrict to $\mathbb{P}^2$. For the braid arrangement A_3, we obtain a map to $\mathbb{P}^5$, whose image is a rational surface, with Hilbert polynomial (compute!)

Let X be the blowup of $\mathbb{P}^2$ at $sing(\mathcal{A})$, and

$$D_{\mathcal{A}} = (n-1)E_0 - \sum_{p_i \in L_2(\mathcal{A})} \mu(p_i)E_i.$$

The intersection pairing on X is given by
$E_0^2 = 1$, $E_{i\neq 0}^2 = -1$ and $E_i \cdot E_{j\neq i} = 0$
Since $K_X = -3E_0 + \sum E_i$, we have

$$\begin{array}{rcl} D_{\mathcal{A}}^2 & = & (n-1)^2 - \sum_{p\in L_2(\mathcal{A})} \mu(p)^2 \\ -D_{\mathcal{A}}K & = & 3(n-1) - \sum_{p\in L_2(\mathcal{A})} \mu(p), \end{array}$$

Proudfoot-Speyer (CM) and Riemann-Roch:

$$\begin{array}{rcl} H^0(D_{\mathcal{A}}) & = & \frac{(n-1)^2 - \sum \mu(p)^2 + 3(n-1) - \sum \mu(p)}{2} + 1 \\ & = & \binom{n+1}{2} - \sum_{p\in L_2(\mathcal{A})} \binom{\mu(p)+1}{2}. \end{array}$$

Double count edges between $L_1(\mathcal{A})$ and $L_2(\mathcal{A})$:

$$\binom{n}{2} = \sum_{p\in L_2(\mathcal{A})} \binom{\mu(p)+1}{2},$$

hence $h^0(D_{\mathcal{A}}) = n$.

Definition 33 *Let $3 \leq k \in \mathbb{Z}$. A k-net in $\mathbb{P}^2$ is a pair $(\mathcal{A}, Z)$ where $\mathcal{A}$ is a finite set of distinct lines partitioned into k subsets $\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{A}_i$ and Z is a finite set of points, such that:*

- *for every $i \neq j$ and every $L \in \mathcal{A}_i$, $L' \in \mathcal{A}_j$, $L \cap L' \in Z$.*
- *for every $p \in Z$ and every $i \in \{1, \ldots, k\}$, $\exists$ a unique $L \in A_i$ containing Z.*

Falk, Libgober, Pereira, Yuzvinsky resonance (next talk!) via nets. Let $m = |\mathcal{A}_i|$ (all equal).

The existence of a (k, m) net

$\rightarrow D_{\mathcal{A}} = A + B$ with $h^0(A) = 2$

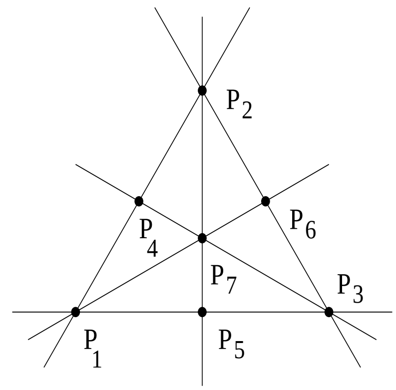
$\rightarrow I_{\mathcal{A}} \supseteq 2 \times 2$ minors $2 \times \left(km - \binom{m+1}{2}\right)$ matrix

$\rightarrow$ Eagon-Northcott complex

$$\cdots \rightarrow S_2(S^2)^* \otimes \Lambda^4 G \rightarrow (S^2)^* \otimes \Lambda^3 G \rightarrow \Lambda^2 G \rightarrow \Lambda^2 S^2 \rightarrow S/I_2 \rightarrow 0.$$

is subcomplex of resolution of $S/I_{\mathcal{A}}$, $G = S(-1)^{km - \binom{m+1}{2}}$

Example 34 For the arrangement A_3



$Z =$ triple points gives a $(3,2)$ net, with $A_i =$ lines thru p_{i+3}, $i = 1,2,3$.

$$A = 2E_0 - \sum_{\{p|\mu(p)=2\}} E_p$$

$$B = 3E_0 - \sum_{p \in L_2(\mathcal{A})} E_p.$$

So $n - \binom{m+1}{2} = 6 - 3 = 3$ and I contains the 2×2 minors of a 2×3 matrix, whose resolution we saw at start of the talk! $D_{\mathcal{A}}$ *almost* gives a De-Concini-Procesi wonderful model: proper transforms of lines are contracted to points.

§**Compactifications**

Fulton-MacPherson $F(X, n)$ combinatorics A_n.
De Concini-Procesi wonderful model for subspace complements (X easy, comb. complex).

$$M(\mathcal{A}) \longrightarrow \mathbb{C}^\ell \times \prod_{D \in G} \mathbb{P}(\mathbb{C}^\ell / D).$$

Version for a lattice L: **Feichtner-Kozlov**.

Definition 35 Building set*:* $G \subseteq L \mid \forall x \in L$*,*

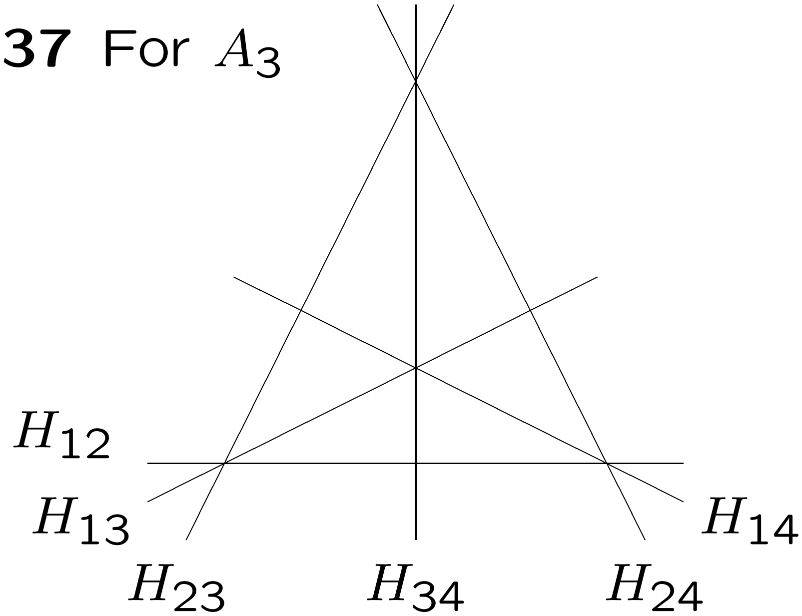
$\max\{G_{\leq x}\} = \{x_1, \ldots, x_m\}$ *has* $[\hat{0}, x] \simeq \prod_{j=1}^{m} [\hat{0}, x_j]$

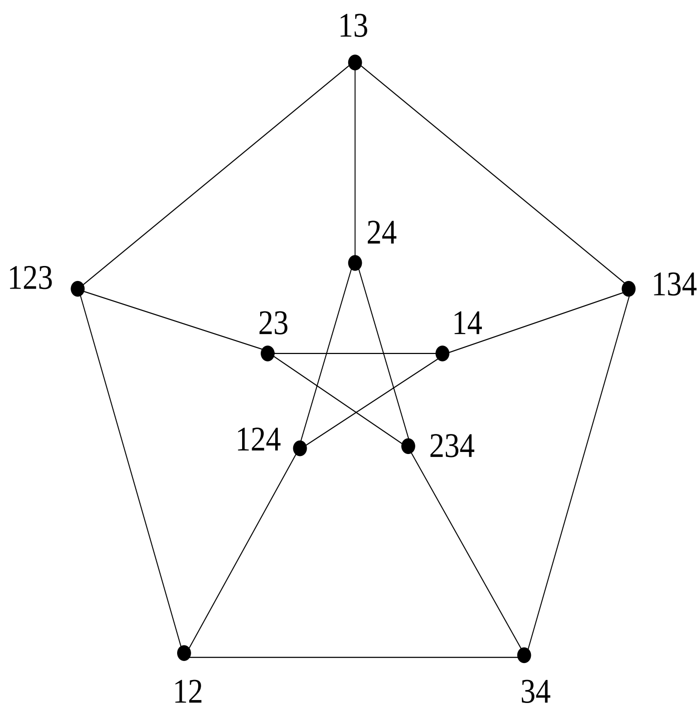
A building set contains all irreducible $x \in L$*.*

Definition 36 $N \subseteq G$ *is* nested *if for any set of incomparable* $\{x_1, \ldots, x_p\} \subseteq N$ *with* $p \geq 2$*,* $x_1 \vee x_2 \vee \cdots \vee x_p$ *exists in* L*, but is not in* G*.*

Nested sets form a simplicial complex $N(G)$, vertices = elements of G (vacuously nested).

Example 37 For A_3





$(12), (123)$ is an edge because there are no incomparable subsets with ≥ 2 elts.

Feichtner and Yuzvinsky G building set in atomic lattice L.

$$D(L,G) = [x_g | g \in G]/I,$$

where I is generated by

$$\prod_{\{g_1,\dots,g_n\} \notin N(G)} x_{g_i} \text{ and } \sum_{g_i \geq H \in L_1} x_{g_i}$$

Theorem 38 *If $\mathcal{A}$ is a hyperplane arrangement and G a building set containing $\hat{1}$, then*

$$D(L,G) \simeq H^*(Y^{\mathbb{P}}_{\mathcal{A},G}, \mathbb{Z}),$$

where $Y^{\mathbb{P}}_{\mathcal{A},G}$ is the wonderful model arising from the building set G.

Importance is that $\overline{M_{0,n}} \simeq Y^{\mathbb{P}}_{A_{n-2},G}$, giving beautiful description of $H^*(\overline{M_{0,n}}, \mathbb{Z})$ (also Knudson, Keel) Compute $H^*(\overline{M_{0,5}}, \mathbb{Z})$.

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