CHAPTER 9

Geometry as Transforming Shapes
Overview

• How are translations, reflections, and rotations related?

• How might you describe the translation, reflection, or rotation of a three-dimensional figure in space?

• How are the operations translation, reflection, and rotation like the operations addition, subtraction, multiplication, and division?
Translations
Translations

In geometry, a translation involves moving (sliding) a shape, without rotating or flipping it.
The shape still looks exactly the same, just in a different place.
A translation is a special kind of congruence transformation.
These design show translations.
Let’s say you were talking on the phone to a friend and you wanted to describe the translation of the triangle CAT from its original location to its new location.

The original triangle is CAT; its translation image is $\triangle C'A'T'$. How would you do this?
Understanding Translations

One common response is to say that we slide the triangle from one location to the other. The mathematical word for slide is *translation*, and we talk about a figure and its *image*.

Various Ways to Describe Translations:
1. Taxicab
2. New Notation
3. Specify the Distance and the angle
4. Vectors
Some of the perspectives are easier to describe if we draw the two triangles on graph paper.
Translations: Taxicab

When we translate a figure, every point on the figure moves the same distance and in the same direction.

*Taxicab Language*

Just as we could ask a cabbie to “go three blocks east and four blocks north,” we could say that each point has been moved 3 units to the right and 4 units up.
Invent New Notation

We can use notation to express this same idea more succinctly:

$$(x, y) \rightarrow (x + 3, y + 4)$$

This notation gives the directions “go 3 units to the right and 4 units toward the top of the page” very succinctly.
Distance and Angle

The figure has been moved a distance of 5 units at a 54-degree angle from the x-axis. That is, C and C’ are 5 units apart; the distance between any point on triangle CAT and the corresponding point on triangle C’A’T’ is 5 units.

Similarly, the angle formed by the rays CC’ and CT is equal to approximately 54 degrees.
Using Vectors

Another alternative is to use a vector. By definition, a **vector** has a length and a direction. For example, the instructions for the translation could be shown by drawing one vector as shown below.

*We will formally define **translation** as a transformation on a plane determined by moving each point in the figure the same distance in the same direction.*
Properties of Translations
Properties of Translations

Now let us determine some of the properties of translations. Earlier we know that two points determine a line. Therefore, if we connect each vertex in the triangle \(TAR\) to its image, \(T'A'R'\), we have three line segments: \(TT'\), \(AA'\), \(RR'\).

The three line segments are all congruent (same length) and all parallel (same direction).
Example of Translating using a Vector

Translate the rectangle in using the translation vector shown.
Translating using a Vector

We can draw four lines, each going through a vertex, that are the same length and parallel to the translation vector. If we copy the length of the vector using a compass, we can quickly mark off the same lengths on the lines to determine the vertices of $M'A'T'H'$. 
Reflections
A **reflection** is a transformation that maps a figure so that a line, called the **line of reflection**, is the perpendicular bisector of every line segment joining a point on the figure and the corresponding point on the reflected figure.
Below is a trapezoid that has been reflected (flipped) in three different ways.

In each case, bold lines denote the original figure and dotted lines denote the flipped image.
Properties of Reflections

True for all reflections: if we connect any point on the original figure with the corresponding point on the reflected figure, the line of reflection is the perpendicular bisector of that line segment. That is, $A\overline{X} = XA$, and $\overline{AA'}$ and line $l$ are perpendicular.
A rotation is a transformation on a plane determined by holding one point fixed and rotating the plane (in our case, the paper) about this point by a certain number of degrees in a certain direction. The fixed point is called the center of rotation.
In order to rotate any figure (in a plane), we need to select a *center of rotation* (that is, the point that does not move), and we need to decide how much to rotate the figure. “How much” is determined by specifying an angle.

Look at the figures and see whether you can guess the center of rotation and the degree of rotation.
These three transformations—translation, reflection, and rotation—are known as **congruence transformations** because the images are congruent to the original figure.
Describe the transformation that has moved the dark pentomino to the position of the light pentomino.

Discussion:

a. The pentomino was **reflected** across a horizontal line below the pentomino.
b. The pentomino was **rotated** 75 degrees counterclockwise.

c. The pentomino was **translated** in a diagonal direction.
d. The pentomino was **rotated** 180 degrees.

![Image of pentomino rotated 180 degrees]


e. The pentomino was **reflected** across a line making a 45-degree angle with a horizontal line.

![Image of pentomino reflected across a 45-degree angle]
f. There is no single translation, reflection, or rotation that will accomplish this move. You could accomplish it in two moves by first **translating** the figure to the right and then **reflecting** across a horizontal line between the two figures. The translation and the reflection line must be parallel.
Combining Slides, Flips, and Turns
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We call any combination of transformations a **composite transformation**.

There is a special name for the composite transformation of a translation followed by a reflection with the condition that the translation vector and the line of reflection are parallel (that is, point in the same direction).

We call such a composition a **glide reflection**.
Glide Reflection

Below a flag is being translated to position 2 and then being reflected across a horizontal line (which is parallel to the translation vector).

The transformation of the flag from position 1 to position 3 is a glide reflection.
Combining Slides, Flips, and Turns

Note that the translation vector does not need to be horizontal—it can be in any direction.

*But whatever the direction of the translation vector, the reflection line must be in the same direction.*

Consider an example of a translation followed by a reflection that is not a glide reflection.
Combining Slides, Flips, and Turns

Below a flag is being translated to position 2 and then being reflected across a vertical line that is *not* parallel to the translation vector.

The composite transformation of the flag is not a glide reflection, because the translation vector and the reflection line are not parallel.
Combining Slides, Flips, and Turns

Probably the most famous example of glide reflection—and one that has helped many students get the sense of glide reflections—is a diagram of footprints on the sand.
Congruence
At the most basic level, we can say that two figures are congruent if they coincide when we superimpose one on the other.

Now that we know that we can move any figure to any position through some combination of transformations, we can give a more general definition of congruence:

Two figures are **congruent** iff there is a translation, reflection, rotation, or glide reflection that maps one figure onto the other.
Transformations and Art

Do you see translations, reflections, and/or rotations in these figures?

Parquet

Churn Dash