Math 241: Honors Homework # 5

Due Wednesday, 20 November 2019 at the beginning of class

For this homework, you have two options: either solve Item 1 only, or solve both Items 2 and 3 only. You can only choose one of these options. (Please do NOT hand in solutions for both options.)

In writing your solutions, fully explain all the important steps. Use full and grammatically correct English sentences. Be clear and concise.

1. This problem gives one way to prove that

\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.
\]

(a) Prove that

\[
\int_0^1 \int_0^1 \frac{1}{1 - xy} \, dx \, dy = \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

(Assume that you may interchange the order of summation and integration. Question: In what?)

(b) Make a change of variables

\[
x = \frac{u - v}{\sqrt{2}}, \quad y = \frac{u + v}{\sqrt{2}}
\]

to evaluate the double integral in (a). To determine the corresponding region in the \(uv\)-plane, you may use the fact that this change of variables is rotation about the origin by an angle of \(\pi/4\). In other words, we get the \(xy\)-plane by rotating the \(uv\)-plane counterclockwise by \(\pi/4\) radians.

Hint: After changing variables, evaluate the \(v\)-integral first. Then, in evaluating the resulting \(u\)-integral, use trigonometric substitution and the identity

\[
\frac{1 - \sin \theta}{\cos \theta} = \sqrt{\frac{1 - \cos(\frac{\pi}{2} - \theta)}{1 + \cos(\frac{\pi}{2} - \theta)}},
\]

which holds for \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\). This will enable you to use the half-angle identities.
2. Find the maximum value of
\[ \oint_C (2y^3 - y) \, dx - x^3 \, dy \]
as \( C \) varies over all positively oriented simple closed curves in \( \mathbb{R}^2 \).

3. Let \( B_4 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 \leq 1\} \) be the unit ball in \( \mathbb{R}^4 \). Calculate its 4-dimensional volume, which is defined by the quadruple integral
\[ \iiint_{B_4} 1 \, dx \, dy \, dz \, dw. \]

(Hint: Write this quadruple integral as an iteration of double integrals, and use what you know about the area of a circle.)