For this homework, you have two options: either solve Item 1 only, or solve both Items 2 and 3 only. You can only choose one of these options. (Please do NOT hand in solutions for both options.)

In writing your solutions, fully explain all the important steps. Use full and grammatically correct English sentences. Be clear and concise.

1. The circle \( x^2 + y^2 = 2y \) is enclosed by the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]
What is the smallest possible area of the ellipse? (Recall that if \( a \) and \( b \) are positive then the area of the ellipse is \( \pi ab \).)

2. Let \( L \) be a positive real number. Prove that, among all triangles with perimeter \( L \), the triangle with the largest area is an equilateral triangle. \((Hint: You can use a result known as Heron’s formula.)\)

3. (a) Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function whose derivative is always positive. It can be proved that it has an inverse function \( f^{-1} \), and this inverse function is strictly increasing and continuous (recall that the inverse function is a function that satisfies \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \)). Use this fact and the definition of derivative to prove that \( f^{-1} \) is differentiable and
\[
\frac{d}{ds} f^{-1}(s) = \frac{1}{f'(f^{-1}(s))}.
\]

(b) Let \( r : \mathbb{R} \to \mathbb{R}^n \) be a vector function that describes a smooth curve that does not intersect itself. Let \( v(s) \) be the reparametrization of the curve with respect to arc length
measured from the point \( r(0) \) in the direction of increasing \( t \). This means we define \( v(s) \) by

\[
v(s) = r(t),
\]

where \( t \) is the unique real number such that

\[
s = \int_0^t |r'(u)| \, du,
\]

if it exists. Prove that

\[
\int_0^\alpha |v'(s)| \, ds = \alpha
\]

for all real numbers \( \alpha \) for which \( v(\alpha) \) is defined. (For convenience, you may assume that \( v(\alpha) \) is defined for all real numbers \( \alpha \). This shows that the reparametrization \( v(s) \) with respect to arc length indeed obeys the condition that the arc length of \( v(s) \) from \( s = 0 \) to \( s = \alpha \) is \( \alpha \).)