CURVATURE BOUNDS FOR WARPED PRODUCTS OF METRIC SPACES

STEPHANIE B. ALEXANDER AND RICHARD L. BISHOP

1. Introduction

1.1. Main theorems. This paper gives sharp conditions for a warped product of metric spaces to have a given curvature bound in the sense of Alexandrov. Thus we have a broad new construction of spaces with curvature bounds, either above (CBA) or below (CBB). As applications, we extend the standard cone and suspension constructions of spaces with curvature bounds [Be, BGP], introduced by Berestovskii, from 1-dimensional base to arbitrary base. We also extend Perelman's CBB doubling theorem [Pm] and Reshetnyak's CBA gluing theorem [R] from 0-dimensional fiber to arbitrary fiber. The proof studies the analytic behavior of billiard trajectories that approximate warped product geodesics, and the geometry of generalized cone points (vanishing points of the warping function).

In Riemannian geometry, warped products are an important source of constructions and counterexamples under curvature constraints [Ps], and we expect even more benefits in the singular setting. For instance, the main construction in Ancel and Guilbault's paper [AG], proving that the interiors of compact contractible n-manifolds ($n \geq 5$) are hyperbolic, when expressed in terms of warped products becomes an example of Theorem 1.1. Indeed, this theorem answers negatively a conjecture in [AG].

For $K \in \mathbf{R}$, we say a continuous function $f : B \to \mathbf{R}$ is *$\mathcal{F}K$-convex* (resp. *$\mathcal{F}K$-concave*) if its restriction to every unitspeed geodesic satisfies the differential inequality $f'' + Kf \geq 0$ (resp. ≤ 0) in the barrier sense (see §2.2). Thus $\mathcal{F}0$-convexity ($\mathcal{F}0$-concavity) is usual convexity (concavity).

Theorem 1.1. (CBA) *Let B and F be complete CAT(K) and CAT(K_F) spaces, respectively. Let $f : B \to \mathbf{R}_{\geq 0}$ be $\mathcal{F}K$-convex, where f is Lipschitz on bounded sets or B is locally compact. Set $X = f^{-1}(0)$.*

1. *If $X = \emptyset$, suppose $K_F \leq K(\inf f)^2$.*
2. *If $X \neq \emptyset$, suppose $f'(0^+)^2 \geq K_F$ at footpoints of d_X-minimizers in B, and $K_F \leq K f(p)^2$ for points $p \in B$ further than $\pi/2\sqrt{K}$ from X.*

Then $B \times_f F$ is CAT(K).

Here $\pi/2\sqrt{K} = \infty$ if $K \leq 0$.

1991 *Mathematics Subject Classification.* 53C20.

Key words and phrases. Alexandrov spaces, CAT(K) spaces, warped products, billiards.

The research described in this paper was made possible in part by Award No. RM1-2381-ST-02 of the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF). Also, important parts were developed while the authors were guests of IHES in Bures-sur-Yvette, France.

Theorem 1.2. (CBB) *Let B and F be complete, finite-dimensional Alexandrov spaces with CBB by K and K_F respectively. Let $f : B \to \mathbf{R}_{\geq 0}$ be an $\mathcal{F}K$-concave, locally Lipschitz function satisfying the boundary condition (†) below. Set $X = (f|\partial B)^{-1}(0)$.*

1. *If $X = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $X \neq \emptyset$, suppose $K_F \geq 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in X$.*

Then $B \times_f F$ has CBB by K.

Here Df_p is regarded as a function on the direction space of B at p. In Theorem 1.2, f necessarily vanishes only at boundary points if f is not identically 0 (§3).

Conversely, suppose a warped product $B \times_f F$ of metric spaces has a curvature bound of K, above or below. Then B inherits that curvature bound because its images in $B \times_f F$ are totally convex. We can also derive $\mathcal{F}K$-convexity or $\mathcal{F}K$-concavity of the warping function f. It remains to show that F has a curvature bound K_F satisfying conditions (1) and (2). We can verify this claim for CBB, but not yet in all CBA cases. It is clearly true for both CBA and CBB if f takes a positive minimum $f(p)$, since then $\{p\} \times F$ is totally convex.

The boundary condition (†) in Theorem 1.2 is the following. It is trivially satisfied if $\partial B = \emptyset$ or if f vanishes on ∂B. Its necessity can be verified; a key fact is that if $B \times_f F$ has CBB by K, so does its double.

(†) *If $B^\dagger$ is the result of gluing two copies of B on the closure of the set of boundary points where f is nonvanishing, and $f^\dagger : B^\dagger \to \mathbf{R}_{\geq 0}$ is the tautological extension of f, then $B^\dagger$ has CBB by K and $f^\dagger$ is $\mathcal{F}K$-concave.*

For example, let $\widetilde{B}$ be the unit ball in $\mathbf{R}^n$, and $\widetilde{f} = d_{\partial \widetilde{B}} : \widetilde{B} \to \mathbf{R}_{\geq 0}$ be distance to $\partial \widetilde{B}$. If B is a convex body in $\widetilde{B}$, set $f = \widetilde{f}|B$. Then f satisfies (†) for $K = 0$ if and only if every ray from the origin that enters B remains there until it hits $\partial \widetilde{B}$. For example, a closed half-ball of $\widetilde{B}$ will do, or more generally, the smaller of the two parts into which $\widetilde{B}$ is cut by any intersecting hyperplane, but not the larger part.

1.2. **Applications.**

The following are examples of new constructions of spaces with curvature bounds above or below. In the first table, B is any CAT(K) space and F is any CAT(K_F) space. In the second table, B and F are any complete, finite-dimensional spaces of CBB by K and K_F respectively. We take $K, K_F \in \{0, \pm 1\}$ by rescaling both B and the warping function. A subset is *ℓ-convex* (an *ℓ-CS*) if it contains, for every pair of its points, all geodesics of length less that ℓ joining them. For example, the zero set of a nonnegatve $\mathcal{F}1$-convex function is a π-CS.

To verify the tables, we must check that the choices of K, K_F and f listed there satisfy the conditions of Theorems 1.1 and 1.2. For case (3B), the concavity of f is a theorem of Perelman: the distance to the boundary of a complete, finite-dimensional Alexandrov space of CBB by 0 is concave [Pm]. The remaining cases of $\mathcal{F}K$-convexity or $\mathcal{F}K$-concavity of f are discussed in [AB2] (also see [G3]). (5A) satisfies condition (2) of Theorem 1.1 because $K_F = Kf(p)^2 = 1$ if p is further than $\pi/2$ from the π-CS. Let us check the boundary condition (†) of Theorem 1.2 in the case (1B). Here, $B^\dagger$ is the double of B, and hence has CBB by $K = -1$ by the doubling theorem [Pm]. The function $f^\dagger : B^\dagger \to \mathbf{R}_{\geq 0}$ that agrees with $f = \cosh d_p^B$

$B \times_f F$ **is CAT**(K)			
	K	K_F	$f : B \to \mathbf{R}_{\geq 0}$
(1A)	-1	-1	cosh (distance to a point)
(2A)	-1	0	exp (Busemann function)
(3A)	0	1	distance to a CS
(4A)	-1	1	sinh (distance to a CS)
(5A)	1	1	sin (min (distance to a π-CS, $\frac{\pi}{2}$))

$B \times_f F$ **has CBB by** K			
	K	K_F	$f : B \to \mathbf{R}_{\geq 0}$
(1B)	-1	-1	cosh (distance to a point)
(2B)	-1	0	exp (Busemann function)
(3B)	0	1	distance to ∂B
(4B)	-1	1	sinh (distance to ∂B)
(5B)	1	1	sin (distance to ∂B)

on each copy of B may be written $f^\dagger = \min\{\cosh d_p^{B^\dagger}, \cosh d_{p^\dagger}^{B^\dagger}\}$, where $p^\dagger$ is the doublepoint of p. In a space of CBB by -1, cosh(distance to a point) is $\mathcal{F}(-1)$-concave [AB2]. Since the minimum of two $\mathcal{F}(-1)$-concave functions is $\mathcal{F}(-1)$-concave, so is $f^\dagger$, as desired. All the remaining conditions are immediate.

These tables extend the following constructions of spaces with curvature bounds from interval base to arbitrary base (we use the terminology of [BGP]):

(1A,B) *hyperbolic cone* : $\mathbf{R} \times_{\cosh x} F$
(2A,B) *parabolic cone* : $\mathbf{R} \times_{\exp x} F$
(3A,B) *linear cone* : $\mathbf{R}_{\geq 0} \times_x F$
(4A,B) *elliptic cone* : $\mathbf{R}_{\geq 0} \times_{\sinh x} F$
(5A,B) *spherical suspension* : $[0, \pi] \times_{\sin x} F$

Example 1.3. To illustrate the hypotheses of Theorems 1.1 and 1.2, let B be an interval (which has CBB by K = any constant and is CAT(K)) and $F = S^{n-1}$ (which has CBB by $K_F = 1$ and is CAT(1)):

(a) Take $B = [1, \infty)$ and $f(x) = x$, so $B \times_f F = (\mathbf{R}^n - \text{unit ball})$. Theorem 1.1 applies for $K = 1$, since $K_F = K(\inf f)^2 = 1$, but not for $K = 0$. Theorem 1.2 does not apply for any choice of K since f cannot satisfy (†); specifically, the tautological extension of f to the double of B has the form of an absolute value function and cannot be $\mathcal{F}K$-concave. Correspondingly, ($\mathbf{R}^n$ − unit ball) is CAT(1); does not have CBA by 0 when $n > 2$, since the fiber at $x = 1$ is a totally geodesic unit sphere; and does not have curvature bounded below.

(b) Take $B = \mathbf{R}$, $K = 0$, and $f(x) = \{0 \text{ if } x \le 0;\ x \text{ if } x \ge 0\}$. Since $f'(0^+) = 1$, Theorem 1.1 applies. $B \times_f F$ is $\mathbf{R}^n$ with the half-line $(-\infty, 0]$ glued to it by identification of the two origins, a space with CBA by $K = 0$.

Example 1.4. If $f(p) = 0$, then in $B \times_f F$ the fiber at p is collapsed to a point. Thus "an F's worth" of copies of B are identified on the zero set of f.

(a) Suppose, as is common, we regard two points at distance $\pi/\sqrt{K}$ as an Alexandrov space of CBB by K, and any discrete set of points at infinite pairwise distances as a CAT(K) space. Then the constructions (3B), (4B) and (5B) of the tables with the first choice of fiber say that CBB by K is preserved by gluing two copies of a space along the boundary; this is Perelman's doubling theorem [Pm]. The constructions (3A), (4A) and (5A) with the second choice of fiber say that CBA by K is preserved by gluing multiple copies of a space along a totally convex set; this is a case of Reshetnyak's gluing theorem [R].

(b) Constructions (3B), (4B) and (5B) yield spaces without boundary if the fiber has no boundary. For example, if B is a convex body in $\mathbf{R}^{n-1}$ and $f : B \to \mathbf{R}_{\ge 0}$ is distance to ∂B, then $B \times_f S^1$ is homeomorphic to S^n. If B is a triangular region in the plane, the metric structure of $B \times_f S^1$ is realized by gluing three solid Euclidean spindles, obtained by rotating the region closest to each edge about that edge, along adjacent pairs of boundary cones.

It follows from Theorems 1.1 and 1.2 that the tangent cone of the warped product $B \times_f F$ in those theorems is given as follows:

$$C_{(p,\phi)}(B \times_f F) = \{C_pB \times_{Df_p} F, \text{ if } f(p) = 0;\ C_pB \times C_\phi F, \text{ if } f(p) > 0\}.$$

Thus the direction space $\Sigma_{(p,\phi)}$ (= unit tangent vectors) is $\Sigma_pB \times_{Df_p} F$ if $f(p) = 0$, and the spherical join $\Sigma_pB \star \Sigma_\phi F$ if $f(p) > 0$.

Example 1.5. (a) In construction (3A) of the tables, take $B = \mathbf{R}^2$ with the origin 0 as CS, so $f = d_0$. If $F = (-\pi/2, \pi/2)$, then we have a smooth warped product away from 0 with Riemannian metric $d\rho^2 + \rho^2(d\theta^2 + d\phi^2)$ for $\phi \in F$. This is in contrast to the Euclidean metric given in spherical coordinates by $d\rho^2 + \rho^2(\sin^2\phi\, d\theta^2 + d\phi^2)$. The direction space of $\mathbf{R}^2 \times_{d_0} S^1$ at 0 is the flat torus, with metric $d\theta^2 + d\phi^2$.

(b) In (3A), take $B = \mathbf{R}^2$ with a triangular region T as CS, so $f = d_T$. Let $F = S^1$. The direction space of $\mathbf{R}^2 \times_{d_T} S^1$ at an edgepoint of the triangle ∂T is S^2 with a segment of length π attached by its endpoints to the two poles. At a vertex of ∂T with internal angle α, the direction space is a cylinder of height $\pi - \alpha$ over S^1, capped by two hemispheres and with a segment of length α attached by its endpoints to the two poles. Note that this long capped cylinder with a short segment attached is indeed a CAT(1) space, as our theorem demands.

Our work illuminates Ancel and Guilbault's main construction in [AG]. Recast in the language of warped products, their goal is a CAT(-1) metric on $B \times \mathbf{R}$, where $B = \mathbf{R}_{\ge 0} \times_{\sinh x} F$ is an elliptic cone over a CAT(1) space F. This must be done so each leaf $B \times \{t\}$ is totally geodesic and isometric to B. The first table solves this problem: namely, B has CBA by -1 by (4A), and therefore $B \times_{\cosh d_p} \mathbf{R}$ has the desired properties by (1A), where d_p is distance in B to a point p. (The leaf $\{p\} \times \mathbf{R}$ is also totally geodesic because p is a minimum point of f.) We have verified that this space is isometric to the one constructed in [AG]. It is conjectured in [AG] that if B is not a cone but a general negatively curved space, then there is no way to put a negatively curved metric on $B \times \mathbf{R}$ so the leaves of the first factor are totally geodesic. To the contrary, by (1A), $B \times_{\cosh d_p} \mathbf{R}$ is such a construction for any $p \in B$.

For other indicators concerning applications of singular warped products, see [CC] and [HLS].

1.3. **Previous work.** The first theorems on warped products of singular spaces were due to Chen [C1], [C2], who proved the case of Theorem 1.1 for which $K_F = 0, f > 0$ and B is 1-dimensional (a tree). Chen's method reduces this case to $\mathbf{R} \times_f \mathbf{R}^2$, so f may be approximated by smooth convex functions and the smooth sectional curvature formula applies. Note that for base spaces B that are not 1-dimensional, no such approximation is available. In [AB1] we proved that Hadamard spaces are closed under warped product with positive convex warping function, namely, the case of Theorem 1.1 with $K = K_F = 0, f > 0$. This is the only case where conditions (1) and (2), and hence the first order conditions on the warping function to which they are equivalent, are invisible. They become visible, for example, in constructing Hadamard spaces using positively curved fibers (one need only think of polar coordinates to see that this is important). As far as we know, there is no previous work on warped products of singular CBB spaces.

1.4. **Outline of paper.** In §3, conditions (1) and (2) of Theorems 1.1 and 1.2 are shown to be equivalent to pointwise first order conditions that are analogues of the Riemannian sectional curvature ones. The proofs of Theorems 1.1 and 1.2 proceed by reducing to the case where the fiber F is the model space S of curvature K_F, and from there to the 1-dimensional fiber case: $B \times_f S = (B \times_f I) \times_\Phi J$, where $\Phi(p, \theta) = f(p) \operatorname{cs}_{K_F}(\theta)$ and cs_{K_F} is a cosine, cosh or constant function. Thus we need the $\mathcal{F}K$-convexity/concavity of Φ on $B \times_f I$, which turns out to be the content of conditions (1) and (2). Note that by definition of a warped product metric, the projection to B of a $(B \times_f I)$-minimizer seeks out lower f values than does a B-minimizer with the same endpoints. The effect is a gain of convexity or loss of concavity, which can be precisely counterbalanced by the second factor of Φ; see §4.

§5 approximates warped products with 1-dimensional fiber by "strip spaces", whose geodesics correspond to billiard paths in a thin-strip billiard table. §6 proves versions of our main theorems that hold when the fiber is R or S^1. §7 reduces the problem to the case where the fiber F is a model space; here, the main concern is the geometry around generalized cone points, i.e., vanishing points of the warping function. As our theorems make clear, many warped products with curvature bounds are forced to have such cone sets.

§8 completes the proof of the main theorems, assuming the convexity of Φ. The latter is proved in §9-11 by analytic estimates on billiard trajectories.

2. Background

2.1. **Alexandrov spaces.** A *CAT(K) space* is a metric space in which any two points at distance less than $\pi/\sqrt{K}$ ($= \infty$ if $K \leq 0$) are joined by a distance-realizing geodesic, and any triangle $\triangle$ of perimeter less than $2\pi/\sqrt{K}$ is *thinner* than its model triangle in the simply conected, 2-dimensional space form S_K of curvature K. That is, the distance between any two points of $\triangle$ is no greater than the distance between the corresponding points on a triangle with the same sidelengths in S_K. We assume the space is intrinsic, but allow infinite distance between points not joined by a rectifiable path. A length space has *CBA by* K if it is locally CAT(K). For CBB by K, inequalities are reversed. An *Alexandrov space of CBB by* K is a complete, locally compact length space satisfying triangle comparisons for triangles in some neighborhood of each point. We shall only consider such spaces of finite dimension [BGP, BBI]. There is no distinction between the local and global triangle comparison properties in complete CBB spaces ([BGP]), or in complete, simply connected spaces with CBA by $K \leq 0$ ([G1], [AB1]; see also [Ba], [BH]). (In the case of CBB by $K > 0$, circles of length $> 2\pi/\sqrt{K}$ and intervals of length $> \pi/\sqrt{K}$ are understood to be excluded.)

Excellent texts and monographs on Alexandrov spaces have recently appeared ([Ba], [BN], [BH], [BBI],[By], [J],[Pl], also see [G2, §1.19$_+$]). We take [BBI], [BH] and [Pl] as references.

Throughout the paper, we always take geodesics to be unitspeed.

2.2. $\mathcal{F}K$**-convexity and** λ**-convexity.** Let $\mathcal{F}K$ denote the family of solutions of the differential equation $f'' + Kf = 0$. By definition, an $\mathcal{F}K$*-convex* function on an intrinsic metric space is one whose restriction to every geodesic satisfies the differential inequality $f'' \geq -Kf$ in the barrier sense. This means that $f \leq g$ if $g \in \mathcal{F}K$ coincides with f at the endpoints of a sufficiently short subsegment. For $\mathcal{F}K$*-concave* functions the inequalities are reversed. See [AB2, G3].

The restriction of an $\mathcal{F}K$-convex or $\mathcal{F}K$-concave function f to a geodesic γ has all the regularity properties of a convex function: left and right derivatives exist at every point (we use f' to denote both) and the second derivative exists almost everywhere. If f is nondecreasing along γ, the $\mathcal{F}K$-convexity ($\mathcal{F}K$-concavity) inequality is equivalent to the statement that $(f')^2 + Kf^2$ is nondecreasing (nonincreasing) along γ.

$\mathcal{F}K$-convexity of f implies local λ*-convexity*, namely, a generalized constant lower bound on f'' along geodesics. Specifically, for continuous functions on an interval, $f'' \geq \phi$ means that $f - F$ is convex where $F'' = \phi$. Equivalently, for each t there is a constant A such that $f(t+\tau) \geq f(t) + A\tau + \phi(t)\tau^2/2 + o(\tau^2)$ [PP, Pl]. The inequality $f'' \geq -Kf$ holds in this sense if and only if it holds in our barrier sense.

If f is a Lipschitz $\mathcal{F}K$-convex function on a CBA space, it follows that the differential Df_p is welldefined on the tangent cone at p, and is linearly homogeneous, Lipschitz, and convex [K]. Moreover, the restriction of Df_p to the direction space (= unit tangent vectors) at p is $\mathcal{F}1$-convex; see the discussion in [AB2]. The same statements, with concavity substituted for convexity, hold in CBB spaces [PP].

In this paper, we regard Df_p as a function on the direction space, whenever that is possible. However, if f is not Lipschitz, Df_p may not be well defined on directions (see [AB2] for an example). In this case, an inequality on $Df_p(v)$ must be read as one on $(f \circ \gamma)'(0)$ for all geodesics γ from p with $\gamma'(0) = v$.

We will use the following Lipschitz approximations:

Lemma 2.1 ([AB2])**.** *Let* $f \geq 0$ *be* $\mathcal{F}K$*-convex on a complete CAT*(K) *space* Y*, where* $K \leq 0$*. Then for every* $A > 0$ *there exists an* $\mathcal{F}K$*-convex function* f^A *that agrees with* f *on* $Y^A = \{p \in Y : (Df)_p \geq -\sqrt{A^2 - Kf(p)^2}\}$ *and satisfies* $\inf(Df^A)_p = -\sqrt{A^2 - Kf(p)^2}$ *if* $p \notin Y^A$*.*

2.3. **Curvature of smooth warped products.** Given a smooth warped product $B \times_f F$, let $\langle\, , \rangle$ denote the warped product metric, and also the metric on B, and

$(,)$ denote the metric on F. The sectional curvature $\mathbf{K}$ of $B \times_f F$, in terms of the sectional curvatures $\mathbf{K}_B$ and $\mathbf{K}_F$, was displayed by Bishop and O'Neill [BO]. We normalize it one more step, by the addition of the condition $\langle x, y\rangle = (v, w) = 0$, which can be imposed on any section Π without loss of generality. Suppose $x + v, y + w$ is such a frame, for $x, y \in T_pB$ and $v, w \in T_{\overline{p}}F$, and let G be the gradient of f. Then

$$\mathbf{K}(\Pi) = \mathbf{K}_B(x,y)\|x\|^2\|y\|^2 - f(p)[\|w\|_F^2 \nabla^2 f(x,x) + \|v\|_F^2 \nabla^2 f(y,y)]$$
$$+f^2(p)[\mathbf{K}_F(v,w) - \|G(p)\|^2]\|v\|_F^2\|w\|_F^2,$$

where $\|\cdot\|, \|\cdot\|_F$ are the norms induced by the Riemannian metric. Thus:

Proposition 2.2. *Let B and F be smooth Riemannian manifolds, with B complete, and let $f : B \to \mathbf{R}_{>0}$ be smooth. Then $B \times_f F$ has CBA (CBB) by K if and only if the following three conditions hold at every point $(p, \overline{p})$:*
(a) f is $\mathcal{F}K$-convex ($\mathcal{F}K$-concave).
(b) $\dim B = 1$ or B has CBA (CBB) by K,
(c) $\dim F = 1$, or for all 2-planes $\Pi_{\overline{p}}$ tangent to F, $\mathbf{K}_F(\Pi_{\overline{p}}) \leq (\geq)\ Kf(p)^2 + \|G(p)\|^2$.

2.4. **Warped products of metric spaces.** Now suppose B and F are intrinsic metric spaces, and $f : B \to \mathbf{R}_{\geq 0}$ is continuous. Distance in the warped product $B \times_f F$ is defined by infimum of path-lengths, where the length of a curve $\gamma = (\gamma_B, \gamma_F)$ for rectifiable curves γ_B and γ_F in B and F is given by:

$$\ell(\gamma) = \int \sqrt{v_B^2(t) + f^2(\gamma_B(t))v_F^2(t)}dt.$$

Here v_B and v_F are the speeds (well-defined almost everywhere, see [BBI, p. 55]) of γ_B and γ_F. Equivalently, $\ell(\gamma)$ is the supremum of the expressions

$$\sum \left(d_B(\gamma_B(t_i), \gamma_B(t_{i-1}))^2 + f(\gamma_B(t_i))^2 d_F(\gamma_F(t_i), \gamma_F(t_{i-1}))^2\right)^{1/2}. \tag{2.1}$$

2.5. **The energy equation.** The *energy equation* for geodesics of a warped product of metric spaces plays a key role in this paper. It is the singular analogue of the smooth phenomenon whereby the projection of a geodesic to the base is a trajectory of the potential function $\frac{1}{2f^2}$. For surfaces of revolution, this is known as Clairaut's theorem. Part (a) below is due to Chen [C1, C2], and the others are found in [AB1].

Lemma 2.3. *For a minimizer $\gamma = (\gamma_B, \gamma_F)$ in $B \times_f F$, $f > 0$,*
(a) γ_F is a minimizer in F.
(b) γ_B is independent of F, except for the total height, i.e., the length of γ_F. Precisely: for another geodesic metric space $\bar{F}$ and minimizer $\bar{\gamma}_F$ in $\bar{F}$ with the same length and speed as γ_F, $(\gamma_B, \bar{\gamma}_F)$ is a minimizer in $B \times_f \bar{F}$.
(c) (Energy equation, version 1) γ_F has speed c_γ/f^2 for some constant c_γ.
(d) (Energy equation, version 2) For some parametrization of γ proportional to arclength, γ_B satisfies $\frac{1}{2}v_B^2 + \frac{1}{2f^2} = E$ almost everywhere, where v_B is the speed of γ_B and E is constant.

3. First order conditions on the warping function

Here we find pointwise first-order conditions on the warping function that are equivalent to the global hypotheses (1) and (2) of our main theorems. Note that the correct conditions are not apparent from the Riemannian case (Proposition 2.2(c)).

Proposition 3.1. **(CBA)** *In Theorem 1.1, an equivalent formulation of conditions (1) and (2) is:*

(3.1) *If* $p \in B - f^{-1}(0)$ *and* $K_F - Kf(p)^2 > 0$, *then* $\inf Df_p \le -\sqrt{K_F - Kf(p)^2}$.

(CBB) *In Theorem 1.2, an equivalent formulation of conditions (1) and (2) is:*

$$K_F - Kf^2 \ge 0, \text{ and } Df_p \le \sqrt{K_F - Kf(p)^2} \text{ for all } p \in B. \tag{3.2}$$

Proof. **(CBA, K ≤ 0)** By $\mathcal{F}K$-convexity, $(f')^2 + Kf^2$ is nondecreasing whenever f is nondecreasing along a geodesic segment. Since we are assuming $K \le 0$, then f is convex and $X = f^{-1}(0)$ is totally convex. We must show that the pointwise condition (3.1) is equivalent to the following global condition:

$$\begin{aligned} &\text{If } X = f^{-1}(0) = \emptyset, \text{ then } K_F \le K(\inf f)^2; \\ &\text{if } X = f^{-1}(0) \ne \emptyset, \text{ then } f'(0^+)^2 \ge K_F \text{ at footpoints of } d_X\text{-minimizers.} \end{aligned} \tag{3.3}$$

Assume (3.1). Then if $K_F - K(\inf f)^2 \ge C > 0$, we have $\inf Df_p \le -\sqrt{C}$ for all $p \in B$. It is easy to conclude, by Zorn's Lemma, that for any $\epsilon > 0$ and any $p \in B - X$, there is a curve from p to X along which $f' \le -\sqrt{C} + \epsilon$. (Such an argument is made in [AB2, Lemma 4.2(1)] .) Therefore $X \ne \emptyset$. Furthermore, if $K_F > 0$, then since $K \le 0$, we may take $C = K_F$. Therefore $f' \le -\sqrt{K_F} + \epsilon$. If follows that $f \ge \sqrt{K_F} d_X$, and this implies the footpoint condition. Thus we have proved (3.3).

Conversely, assume (3.3). If $X \ne \emptyset$, then any $p \in B - X$ has a unique footpoint q on X. Along a minimizer γ with $\gamma(0) = q$ and $\gamma(L) = p$, the value of $(f')^2 + Kf^2$ is at least its initial value, since f is nondecreasing. This initial value is $\ge K_F$ by (3.3). Therefore $f'(L^-)^2 + Kf(p)^2 \ge K_F$. If $K_F - Kf(p)^2 > 0$, this means $\inf Df_p \le -\sqrt{K_F - Kf(p)^2}$, as required.

On the other hand, suppose $X = \emptyset$. By (3.3), if $K = 0$ then $K_F \le 0$ and the condition (3.1) is empty. Assume $K < 0$. By (3.3), $K_F \le K(\inf f)^2$. Assuming $K_F - Kf(p)^2 > 0$, then $f(p) > \inf f$. For $1 \ge \epsilon > 0$ and $\epsilon < f(p)$, choose p_ϵ with $f(p_\epsilon) \le \inf f + \epsilon$. Then $Kf(p_\epsilon)^2 \ge K(\inf f)^2 + \epsilon(2K \inf f + K\epsilon) \ge K_F - C\epsilon$. Along a minimizer γ with $\gamma(0) = p_\epsilon$ and $\gamma(L) = p$, we may assume f is increasing (otherwise replace p_ϵ with a point $\widetilde{p}_\epsilon$ on γ where $f(p_\epsilon) = f(\widetilde{p}_\epsilon)$). The value of $(f')^2 + Kf^2$ is at least its initial value, hence at least $K_F - C\epsilon$. For $\epsilon \to 0$, we again find that $\inf Df_p \le -\sqrt{K_F - Kf(p)^2}$ if $K_F - Kf(p)^2 > 0$.

(CBB, K ≤ 0) We claim that (3.2) is equivalent to:

$$\begin{aligned} &\text{If } X = (f|\partial B)^{-1}(0) = \emptyset, \text{ then } K_F \ge Kf^2; \\ &\text{if } p \in X = (f|\partial B)^{-1}(0), \text{ then } K_F \ge 0 \text{ and } Df_p \le \sqrt{K_F}. \end{aligned} \tag{3.4}$$

By (†), we may assume f vanishes at ∂B, by replacing B with $B^\dagger$ and f with $f^\dagger$. Clearly (3.2) implies (3.4). Conversely, assume (3.4), and f not identically 0. Then f is positive on nonboundary points. Indeed, suppose $f(p) = 0$, and take a minimizer from p to a nonvanishing point of f. By [PP], we may extend this minimizer as a quasigeodesic γ indefinitely. However, if γ reaches a boundary point, let us terminate it there. Then the restriction of f to γ is $\mathcal{F}K$-concave. Indeed, every point p in the interior of B has a neighborhood in which the restrictions of f to geodesics satisfies $f'' \leq -Kf(p) + \epsilon$ in the sense of §2.2. By [PP, 6.1], the same statement extends to quasigeodesics. But then along γ, it is easy to construct a sequence of continuous functions h_ϵ converging uniformly to $-Kf$, such that $f'' \leq h_\epsilon$; this condition is another equivalent formulation of $f'' \leq -Kf$ [PP]. Since $f \geq 0$, it follows that γ does not extend past p in the negative direction, and hence $p \in \partial B$.

Suppose $f(p) > 0$ and $Df_p(v) > 0$. Then there is a quasigeodesic γ in B with initial direction v, where we extend γ in the negative direction indefinitely or until it reaches the boundary. Since the restriction of f to γ is $\mathcal{F}K$-concave, either γ terminates in the negative direction at a point $q \in \partial B$, or else the restriction of f to γ has positive derivative on a maximal interval $(a, 0]$ and $\lim_{t\to a^+} f'(t) = 0$ (where possibly $a = -\infty$). In the first case, by hypothesis we have $K_F \geq 0$ and $f' \leq \sqrt{K_F}$ at q. Since f vanishes at q and $(f')^2 + Kf^2$ is nonincreasing along γ, then $Df_p(v)^2 + Kf(p)^2 \leq K_F$, and so (3.2) holds at p. In the second case, $\lim_{t\to a^+}((f'(t))^2 + Kf(t)^2) = \lim_{t\to a^+} Kf(t)^2 \leq K_F$. The last inequality holds by assumption if $X = \emptyset$, and holds otherwise since $K_F \geq 0$ and $K \leq 0$. Since $(f')^2 + Kf^2$ is nonincreasing along γ, again (3.2) holds at p.

(CBA & CBB, $\mathbf{K > 0}$) We refer the case $K > 0$ (by scaling, we may take $K = 1$) back to the case $K = 0$ by coning. A geodesic metric space B is CAT(1) (has CBB by 1) if and only if its linear cone $C(B)$ is CAT(0) (has CBB by 0) (see [BBI, Th. 4.7.1]). Moreover, $f : B \to \mathbf{R}_{\geq 0}$ is $\mathcal{F}1$-convex ($\mathcal{F}1$-concave) if and only if its homogeneous linear extension $C(f) : C(B) \to \mathbf{R}_{\geq 0}$, namely $C(f)(r,p) = rf(p)$, is convex (concave) [AB2, Lemma 3.5].

We claim that (3.1) for B, f and $K = 1$ is equivalent to the same condition for $C(B), C(f)$ and $K = 0$, and similarly for (3.2). In the first case, this follows by calculating $\inf DC(f)_{(r,p)}$ to be (see [AB2, (4)]):

$$-f(p), \text{ if } \inf Df_p > 0; \ -\sqrt{f(p)^2 + (\inf Df_p)^2}, \text{ if } \inf Df_p \leq 0.$$

In the second case, $\sup_{\{w: DC(f)_{(r,p)}(w) \geq 0\}} DC(f)_{(r,p)}(w)$ is calculated ([AB2, (5)]:

$$f(p), \text{ if } \sup Df_p(v) < 0; \ \sqrt{f(p)^2 + (\sup_{\{v:\, Df_p(v) \geq 0\}} Df(v))^2}, \text{ if } \sup Df_p(v) \geq 0.$$

Here we also need to observe that p is a limit of points q satisfying $\sup Df_q \geq 0$.

It suffices to prove a similar equivalence for (3.3). Note that when $K > 0$, condition (2) of Theorem 1.1 has an additional feature. Thus we need a modification $(3.3)^+$ of (3.3), obtained by adding to it:

$$\text{if } X = f^{-1}(0) \neq \emptyset \text{ and } d_X(p) \geq \pi/2\sqrt{K}, \text{ then } K_F \leq Kf(p)^2.$$

Now we must verify that $(3.3)^+$ for B, f and $K = 1$ is equivalent to (3.3) for $C(B), C(f)$ and $K = 0$. Here $C(f)^{-1}(0) = C(f^{-1}(0)) \cup v$, where v is the vertex of the cone $C(B)$. Thus since $C(f)^{-1}(0) \neq \emptyset$, our (3.3) condition on $C(f)$ is simply: $C(f)'(0^+)^2 \geq K_F$ at footpoints on $C(f)^{-1}(0)$ in $C(B)$. Since geodesics in $C(B)$ project to geodesics in B (Lemma 2.3(a)), such a footpoint either corresponds to a footpoint on $f^{-1}(0)$ in B, or is the vertex v. In the former case, the derivatives of $C(f)$ and f from the footpoints, with respect to unit-speed parameters, are equal. On the other hand, if (r,p) has footpoint v, then $C(f)'(0^+) = f(p)$. Such a minimizer must make angle $\geq \pi/2$ at v with every ray in $C(X)$, and so $d_X(p) \geq \pi/2$. Therefore our $(3.3)^+$ condition implies our (3.3) condition. For the converse, note that if $f^{-1}(0) = \emptyset$, then v is the only footpoint, and we conclude that $K_F \leq f(p)^2$ for all $p \in B$.

The equivalence of (3.4) for B, f and $K = 1$ with (3.4) for $C(B), C(f)$ and $K = 0$ proceeds by the same considerations. Thus the proposition holds for $K > 0$ because it holds for $K = 0$. □

Remark 3.2. (3.2) implies f is locally Lipschitz. Indeed, positive values of $Df_p(v)$ are locally bounded; and for any given negative value of $Df_p(v)$, v is the limit of initial directions of geodesics from p, and so $|Df_p(v)|$ is the limit of $Df_{p_i}(v_i) > 0$ where $p_i \to p$.

4. Constructing convex functions on warped products

The following theorems, which are proved in §9-11, lift convex/concave functions from a base space to its warped product with an interval. We are going to prove our main theorems by reduction to two applications of the 1-dimensional fiber case, and the lifted functions Φ discussed here are the warping functions in the second application. Therefore it is essential to know their convexity/concavity.

Let cs_{K_F} be the solution to the initial-value problem $y'' + K_F y = 0$ with $y(0) = 1, y'(0) = 0$, and I be the interval containing 0 on which $y \geq 0$. On a warped product of the form $B \times_f I$, set

$$\Phi = (f \circ \pi_B) \cdot (\mathrm{cs}_{K_F} \circ \pi_I) : B \times_f I \to \mathbf{R}_{\geq 0},$$

where π_B and π_I are the projections onto the factors.

Theorem 4.1. *Let B be a complete CAT(K) space with $K \leq 0$, $f : B \to \mathbf{R}_{\geq 0}$ be a $\mathcal{F}K$-convex function that is Lipschitz on bounded sets, and K_F be a constant. If $p \in B - f^{-1}(0)$ and $K_F - Kf(p)^2 > 0$, suppose* $\inf Df_p \leq -\sqrt{K_F - Kf(p)^2}$. *Then Φ is $\mathcal{F}K$-convex.*

Theorem 4.2. *Let B be a complete, finite-dimensional Alexandrov space with CBB by $K \leq 0$, $f : B \to \mathbf{R}_{\geq 0}$ be a locally Lipschitz $\mathcal{F}K$-concave function vanishing on ∂B, and K_F be a constant. Suppose $K_F - Kf^2 \geq 0$ and $Df_p \leq \sqrt{K_F - Kf(p)^2}$ for all $p \in B$. Then Φ is $\mathcal{F}K$-concave.*

For a better understanding of the geometric meaning of these theorems, let us identify Φ in the special examples 3(A), 4(A) and 3(B), 4(B) of the tables in §1.2.

Example 4.3. Suppose B is CAT(K) for $K \in \{-1, 0\}$, and take $K_F = 1$. Let X be a convex subset (CS) of B, and take $f = \{d_X$ if $K = 0$; $\sinh d_X$ if $K = -1\}$. Then f is $\mathcal{F}K$-convex (see [AB2]), and satisfies (3.3). Therefore by Proposition 3.1 and Theorem 4.1, Φ is $\mathcal{F}K$-convex on $B \times_f I$, where $I = [-\pi/2, \pi/2]$. Here $\Phi(p, \theta) = f(p) \cos\theta$ for $(p, \theta) \in B \times_f I$. Since $K \leq 0$, then Φ is convex, so the zero set Ξ of Φ is a CS in $B \times_f I$. Specifically, Ξ is the image of $(B \times \{-\pi/2\}) \cup (B \times \{\pi/2\})$ under the identification map $B \times I \to B \times_f I$, and consists of two copies of B glued on the totally convex set X. It can be shown that in this example, $\Phi = \{d_\Xi$ if $K = 0$; $\sinh d_\Xi$ if $K = -1\}$.

Example 4.4. Suppose B has CBB by K for $K \in \{-1, 0\}$, and $\partial B \neq \emptyset$. Take $K_F = 1$. Let $f = \{d_{\partial B}$ if $K = 0$; $\sinh d_{\partial B}$ if $K = -1\}$. Then f is $\mathcal{F}K$-concave [Pm],[AB2], and satisfies (3.4). Therefore by Proposition 3.1 and Theorem 4.2, Φ is

$\mathcal{F}K$-concave on $B \times_f I$, where $I = [-\pi/2, \pi/2]$ and $\Phi(p,\theta) = f(p)\cos\theta$ for $(p,\theta) \in B \times_f I$. The zero set Ξ of Φ is the image in $B \times_f I$ of $(B \times \{-\pi/2\}) \cup (B \times \{\pi/2\})$, and consists of two copies of B identified on their boundary. It can be shown that $\Phi = \{d_\Xi$ if $K = 0$; $\sinh d_\Xi$ if $K = -1\}$. Note that if we knew $B \times_f I$ to have CBB by 0 *a priori*, then we could identify Ξ as its boundary and conclude d_Ξ was concave. In §6, it is convenient to reverse the order and use the concavity of Φ, guaranteed by Theorem 4.2, to prove that $B \times_f I$ has CBB by 0.

5. Strip spaces

5.1. **Metrics on strip spaces.** In [AB1] we approximated $B \times_f \mathbf{R}$ for $K = 0$ by nonpositively curved strip spaces, created from strips in $B \times \mathbf{R}$ with the product metric. Here we briefly review the construction, and show that when $K = -1$, we can retain the desired curvature bound on the strip space by placing a suitable hyperbolically warped metric $B\,{}_{\mathrm{cs}_k}\!\times \mathbf{R}$ $(= \mathbf{R} \times_{\mathrm{cs}_k} B$ with the order of factors reversed) on the strips.

Take isometric copies (indexed by $i \in \mathbf{Z}$) of a *strip* $W^{(i)}$ in $B \times \mathbf{R}$:

$$W_\epsilon^{(i)} = \{(p,u) : -\epsilon f(p) \le u \le \epsilon f(p)\}. \tag{5.1}$$

The *strip space* W_ϵ is obtained by gluing the strips $W_\epsilon^{(i)}$ sequentially along their boundaries. There is a natural homeomorphism $\varphi_\epsilon : B \times_f \mathbf{R} \to W_\epsilon$, given by decomposing $B \times_f \mathbf{R}$ into horizontal strips $\{(p,y) : (2i-1)\epsilon \le y \le (2i+1)\epsilon\}$. The restriction $\varphi_\epsilon^{(i)}$ of φ_ϵ to this horizontal strip maps it onto $W_\epsilon^{(i)}$ by linear reparametrization of the fibers:

$$\varphi_\epsilon^{(i)} : (p,y) \mapsto (p, f(p)[y - 2i\epsilon]). \tag{5.2}$$

Take the metric of W_ϵ to be that induced from $B\,{}_{\mathrm{cs}_k}\!\times \mathbf{R}$, where k will be chosen to control the curvature of W_ϵ in the limit. In the CBB case, this curvature control depends on Perelman's doubling theorem [Pm], see also Petrunin's more general gluing theorem [Pn]; these theorems concern gluing CBB spaces along isometric boundaries. In the CBA case, we use the dual construction, namely, Reshetnyak's gluing theorem [R] for gluing along isometric totally convex sets. This construction was used previously in [AB1], and a similar construction was used by Burago, Ferleger and Kononenko in [BFK] to solve long-standing billiards problems. We decompose copies of $B\,{}_{\mathrm{cs}_k}\!\times \mathbf{R}$ into three regions, $U_\epsilon^{(i)} \cup W_\epsilon^{(i)} \cup L_\epsilon^{(i)}$, where

$$U_\epsilon^{(i)} = \{(p,u) : \epsilon f(p) \le u\}, \quad L_\epsilon^{(i)} = \{(p,u) : u \le -\epsilon f(p)\}. \tag{5.3}$$

Construct a space W_ϵ^* by sequentially identifying the isometric sets $U_\epsilon^{(i)}$ and $L_\epsilon^{(n+1)}$. Thus W_ϵ^* is our strip space W_ϵ augmented by attaching "fins" $U_\epsilon^{(i)} \cong L_\epsilon^{(n+1)}$. The approximating strip space W_ϵ may itself have positively infinite curvature, but is embedded in W_ϵ^* in such a way that we can bound its curvature in the limit.

Assume $K \in \{-1, 0\}$. If $K = 0$, set $k = 0$. If $K = -1$, set $k = k(\epsilon) = -1 + \epsilon$ in case (CBA), and $k = -1$ in case (CBB).

Lemma 5.1. (CBB) *Let B be a complete, finite-dimensional space of CBB by K with empty boundary, and $f : B \to \mathbf{R}_{>0}$ be a positive $\mathcal{F}K$-concave function. Then the strip space W_ϵ, formed by sequentially gluing strips $W_\epsilon^{(n)} \subset B_{\mathrm{cs}_K} \times \mathbf{R}$, has CBB by K.*

(CBA) *Let B be a CAT(K) space and $f : B \to \mathbf{R}_{\geq 0}$ be a Lipschitz $\mathcal{F}K$-convex function. Then for ϵ sufficiently small, the augmented strip space W_ϵ^*, formed by attaching "fins" to W_ϵ, is CAT(k) for $k = k(\epsilon)$ as above.*

Proof. First note that $B \,_{\mathrm{cs}_k}\!\times \mathbf{R}$ has CBB by k in the case (CBB), and CBA by k in the case (CBA). Such standard coning constructions are discussed, for example, in [BBI]. Now by the doubling and gluing theorems, respectively, we need only show that in the CBB case, the strip $W_\epsilon^{(i)}$ defined by (5.1) is convex in $B \,_{\mathrm{cs}_k}\!\times \mathbf{R}$, and in the CBA case, the fin $U_\epsilon^{(i)}$ defined by (5.3) is convex in $B \,_{\mathrm{cs}_k}\!\times \mathbf{R}$ when ϵ is sufficiently small.

Since minimizers in $B \,_{\mathrm{cs}_k}\!\times \mathbf{R}$ project to minimizers in B (Lemma 2.3 (a)), convexity of the strip $W_\epsilon^{(i)}$ or fin $U_\epsilon^{(i)}$ is equivalent to convexity in the cylinder above a minimizer in B. If $k = 0$, this is immediate from the concavity (CBB) or convexity (CBA) of f along geodesics of B. In the remaining cases, the problem reduces to the hyperbolic plane $\mathbf{R} \,_{\mathrm{cs}_k}\!\times \mathbf{R}$. We must verify the convexity of $\{(t, u) : u \leq \epsilon f(t)\}$, where $f''(t) - f(t) \leq 0$ in the barrier sense, in the case (CBB); and the same statement with the inequalities reversed in (CBA). By definition, the differential inequalities imply that the graph of f lies above (CBB) or below (CBA) sufficiently fine inscriptions by broken $\mathcal{F}(-1)$-affine graphs, and the one-sided tangent vectors of these inscriptions rotate downward (CBB) or upward (CBA) at the breaks. Therefore it suffices for this calculation to take $f \in \mathcal{F}(-1)$ with a given Lipschitz constant.

Writing the metric as $ds^2 = \cosh^2(\sqrt{-k}u)dt^2 + du^2$, we calculate the curvature vector of the curve $u = \epsilon f$. Specifically, if $V = \partial_t + f'(t)\partial u$ is the velocity vector and N is the unit normal, then the ∂_u-component of the curvature vector κN is:

$$
\begin{aligned}
(5.4)\qquad &|v|^{-4}[\cosh^2\sqrt{-k}\epsilon f(\epsilon f'' - \sqrt{-k}\cosh\sqrt{-k}\epsilon f \sinh\sqrt{-k}\epsilon f) \\
&\qquad - 2\epsilon^2 (f')^2 \sqrt{-k}\cosh\sqrt{-k}\epsilon f \sinh\sqrt{-k}\epsilon f].
\end{aligned}
$$

(CBB) The condition for convexity of $W_\epsilon^{(i)}$ is that κN point downward, hence that the expression in (5.4) be nonpositive. After substituting $f'' = f$ and multiplying by $|v|^4/\epsilon f \cosh^2\sqrt{-k}\epsilon f$, we find that concavity is detected by the following inequality:

(5.5)

$$(-k)[(\cosh\sqrt{-k}\epsilon f)(\sinh\sqrt{-k}\epsilon f)/\sqrt{-k}\epsilon f + 2\epsilon^2(f')^2(\tanh\sqrt{-k}\epsilon f)/\sqrt{-k}\epsilon f] \geq 1.$$

Clearly this inequality holds if $k = -1$, since then it holds for the first term and the second term is positive .

(CBA) For convexity of $U_\epsilon^{(i)}$, we want the curvature vector to point upward, hence the inequality in (5.5) to be reversed. Since f' and f are bounded, the lefthand expression is on the order of $k[1 + C\epsilon^2]$ for a uniform constant C. Thus the reversed inequality (5.5) is satisfied for ϵ sufficiently small if $k = k(\epsilon) = -1 + \epsilon$. □

5.2. **Geodesics in strip spaces.** The following lemma is an analogue for strip spaces of the result that the vertical projection of a warped product minimizer is monotonic. We consider a strip space W_ϵ and augmented strip space W_ϵ^*, constructed as in §5.1 from strips (5.1) and fins (5.3) that lie in $B \,_{\mathrm{cs}_k}\!\times \mathbf{R}$ for $k \leq 0$.

Lemma 5.2. *(1) Suppose f is continuous and positive on the geodesic metric space B, and γ is a minimizer in the strip space W_ϵ beginning and ending at midlevels* *of strips. Then γ consists of segments across the interior of strips except for single boundary points connecting each segment to the next.*

(2) In the setting of Lemma 5.1 (CBA), suppose $f > 0$ and γ is a minimizer in the augmented strip space W_ϵ^* beginning and ending at midlevels of strips. Then γ lies entirely in W_ϵ.

Proof. Part (1). Suppose γ had more than one point in a single strip boundary. Then the first point p_1 in that boundary would be preceded by an interior minimizer to a midlevel point p_0, and the last point p_2 in that boundary would be followed by an interior minimizer to a midlevel point p_3 of the next strip. The part between p_1 and p_2 must remain in the union of the two half strips, since otherwise it would intersect some single midlevel twice and a horizontal minimizer segment between those two intersections would be shorter. Now let $\widetilde{\gamma}$ consist of the part of γ already in the first strip, together with the reflection into the first strip of the part of γ in the second strip. Since $\widetilde{\gamma}$ runs from p_0 to p_3' (the copy of p_3 in the first strip), and minimizes in the strip from p_0 to p_2, and also from p_1 and p_3', then $\widetilde{\gamma}$ is a local minimizer in the strip. Since the projection of $\widetilde{\gamma}$ to B is a geodesic, the domain below $\widetilde{\gamma}$ in the half strip is isometric in its intrinsic metric to a domain in the Euclidean or hyperbolic plane. If we extend the latter domain infinitely upward, the complementary upper domain has locally convex boundary because $\widetilde{\gamma}$ is a local minimizer, and hence must be convex. This contradicts the fact that the horizontal segment from p_0 to p_3' is a minimizer but does not lie above $\widetilde{\gamma}$.

Part (2). Suppose γ has a point p in a fin $U_\epsilon^{(0)} = L_\epsilon^{(1)}$ not in W_ϵ, and begins and ends at midlevel points p_0 and p_3 of the adjacent strips $W_\epsilon^{(0)}$ and $W_\epsilon^{(1)}$. Since the fin $U^{(0)}$ is convex, γ travels from p_0 to p_1 in $W_\epsilon^{(0)}$, from p_1 to p_2 in $U_\epsilon^{(0)}$, and from p_2 to p_3 in $W_\epsilon^{(1)}$. Now isometrically copy the subsegment from p_2 to p_3 into $W_\epsilon^{(0)}$ by reflection, to get a curve $\widetilde{\gamma}$ in $W_\epsilon^{(0)} \cup U_\epsilon^{(0)}$ which minimizes between p_0 and p_2, and also between p_1 and p_3' (the copy of p_3). Thus $\widetilde{\gamma}$ is a geodesic in its cylinder, which is isometric to a domain in the euclidean or hyperbolic plane. This is impossible because the unique geodesic from p_0 to p_3' in this cylinder is the horizontal one, not containing p. □

6. Proofs of the main theorems for 1-dimensional fiber

In this section we prove versions of Theorems 1.1 and 1.2 that hold when the fiber is $\mathbf{R}$ or S^1 (the circle of length 2π). We also assume $K \le 0$. In §7 - 8, we derive the general theorems from these special cases.

Theorem 6.1. *Let B be a CAT(K) space for $K \le 0$, and $f : B \to \mathbf{R}_{\ge 0}$ be a K_B-convex function that is Lipschitz on bounded sets.*

1. *Then $B \times_f \mathbf{R}$ is CAT(K).*
2. *Suppose $X = f^{-1}(0)$ is nonempty. Along every d_X-minimizer, suppose $f'(0^+) \ge 1$ at the footpoint. Then $B \times_f S^1$ is CAT(K).*

Proof. Part (1). By Lemma 5.1(CBA), the augmented strip space W_ϵ^* is CAT(K). To conclude that $B \times_f \mathbf{R}$ is CAT(K) we use a pointwise distance-convergence theorem (see [AB1], Theorem 6.1, due to Buyalo [By], and Theorem 6.3): *Suppose that for any $\eta > 0$ there is an η-net A_η in W such that for all $P, Q \in A_\eta$ we have $\lim_{n\to\infty} d(\varphi_\epsilon(P), \varphi_\epsilon(Q)) = d(P,Q)$, where $\epsilon = 2^{-n}$. Then W is CAT(K).* It suffices to work in subsets of W on which $f < M$ for some M, and then we take the η-nets to be the union of midlevel sets $\{(p, i2^{-m}) : p \in B, i \in \mathbf{Z}\} = A_\eta$, where $m \in \mathbf{Z}$ and $M2^{-m} < \eta$.

To verify the hypothesis of this convergence theorem we proceed as in [AB1, Lemma 7.2], except for minor changes. We now outline the procedure. To show that $d(\varphi_\epsilon(P), \varphi_\epsilon(Q))$ is not much greater than $d(P, Q)$, we start by taking a chain of nearly intermediate points along a connecting curve to approximate $d(P, Q)$. It is important to note that that curve may be assumed to have monotonic projection to $\mathbf{R}$. When the strips W_ϵ have the warping function $\mathrm{cs}_k, k < 0$, it is easy to show that the distances in W_ϵ are greater than in the case $k = 0$, but only by a negligible amount as $\epsilon \to 0$. Otherwise the proof is the same. In the other direction, to show that $d(P, Q)$ is not much greater than $d(\varphi_\epsilon(P), \varphi_\epsilon(Q))$, we represent the latter distance by a geodesic in W_ϵ, which is possible for P, Q in our η-nets by Lemma 5.2; this avoids a complication of [AB1, Lemma 7.2] due to the approximation by points at which Df is bounded. (For horizontal curves the two distances are equal, showing that there is no difficulty from points at which $f = 0$.) The distance $d(P, Q)$ is then estimated by the length of the corresponding curve in W, calculated from its velocity. The warping factor $\mathrm{cs}_k, k < 0$, strengthens the desired inequality in this direction. □

Part (2). If $q \in X$, then any two points in the fiber above q in $B \times_f \mathbf{R}$ have distance 0, hence are identified. Thus $\pi_B^{-1}(X) = X$. Lemma 2.3(a) implies that $B \times_f I$ is convex in $B \times_f \mathbf{R}$, for $I = [-\pi/2, \pi/2]$, and hence is $\mathrm{CAT}(K)$ by part (1). Let Ξ be the image in $B \times_f I$ of $B \times \{-\pi/2\} \cup B \times \{\pi/2\}$. We claim that Ξ is convex in $B \times_f I$. It then follows from the gluing theorem that $B \times_f S^1$ is $\mathrm{CAT}(K)$, as desired. Indeed, by Proposition 3.1, our hypotheses imply those of Theorem 4.1 for $K_F = 1$. Therefore the function $\Phi = f(p) \cos\theta$ for $(p, \theta) \in B \times_f I$ is $\mathcal{F}K$-convex, and hence convex since $K \leq 0$. Then the zero set of Φ is totally convex in $B \times_f I$. Since the zero set is Ξ, the proof is complete. □

Theorem 6.2. *Let B be a complete, finite-dimensional Alexandrov space of CBB by $K \leq 0$, and $f : B \to \mathbf{R}_{\geq 0}$ be a locally Lipschitz $\mathcal{F}K$-concave function vanishing on ∂B.*

1. *If $\partial B = \emptyset$, then $B \times_f \mathbf{R}$ has CBB by K.*
2. *Suppose $\partial B \neq \emptyset$. If $Df_q \leq 1$ for all $q \in X$, then $B \times_f [-\pi/2, \pi/2]$ and $B \times_f S^1$ have CBB by K.*

Proof. We may assume f not identically 0, since $B \times_0 F = B$. Then f is positive on nonboundary points (see proof of Proposition 3.1).

Part (1). Since $f > 0$, then by Lemma 5.1(CBB), the strip space W_ϵ has CBB by K. Again, the arguments of [AB1, Lemma 7.2] imply that $B \times_f \mathbf{R}$ has CBB by K. □

Part (2). By Proposition 3.1, our hypotheses imply those of Theorem 4.2 for $K_F = 1$. Therefore the function $\Phi(p, \theta) = f(p) \cos\theta$ for $(p, \theta) \in B \times_f I$ is $\mathcal{F}K$-concave, where $I = [-\pi/2, \pi/2]$. To summarize, $(B - \partial B) \times_f I$ has CBB by K, as in (1). Moreover, Φ is nonnegative and $\mathcal{F}K$-concave on $B \times_f I$, and its zero set is the image of $(B \times \{-\pi/2\}) \cup (B \times \{\pi/2\})$.

The proof is now easily finished if $K = 0$. In that case, Φ is concave. Since its superlevel sets are totally convex, they have CBB by 0 globally. Thus the limit set $B \times_f I$ has CBB by 0, as required.

If $K = -1$, the superlevel sets of Φ are no longer convex, only almost so. It seems that finishing the proof in the context of standard Alexandrov spaces would be rather long and technical. Instead, we give a simple argument in the more general context of *Minkowski cones* and *curvature bounds in the energy sense*, as developed in [AB3]. We only need the following facts, where Y is a geodesic metric space (compare §3, $K > 0$):

- **(a):** Y has CBB by -1 if and only if its Minkowski cone $C_-(Y)$ has CBB by 0 in the energy sense.
- **(b):** $f : Y \to \mathbf{R}_{\geq 0}$ is $\mathcal{F}(-1)$-concave if and only if its homogeneous linear extension $C(f)$ to $C_-(Y)$ is concave.

Definitions and details will be found in [AB3], but it is now formally easy to finish our proof in the case $K = -1$. Namely, since Φ is $\mathcal{F}(-1)$-concave on $B \times_f I$, then by (b), $C(\Phi)$ is concave on $C_-(B \times_f I)$. The zero set of $C(\Phi)$ is $C_-((B \times \{-\pi/2\}) \cup (B \times \{\pi/2\}))$. Since $(B - \partial B) \times_f I$ has CBB by -1, then by (a), $C_-((B - \partial B) \times_f I)$ has CBB by 0 in the energy sense. Therefore the complement of the zero set of $C(\Phi)$ has CBB by 0 in the energy sense. Since the superlevel sets of $C(\Phi)$ are totally convex, their limit set $C_-(B \times_f I)$ also has CBB by 0 in the energy sense. Therefore by (a), $B \times_f I$ has CBB by -1. By the doubling theorem, so does $B \times_f S^1$. □

Remark 6.3. The proofs of our main theorems rely on [AB3] only for the case of Theorem 1.2 with $K = -1$ and $f^{-1}(0) \neq \emptyset$. For instance, in the ten examples of the tables in §1.2, [AB3] was used only in the proof of (4B).

7. Reduction of the fiber

7.1. **Reduction to model space fiber.** Generalized cone points, i.e., vanishing points of the warping function, are our main concern in this section. We show that a curvature bound for $B \times_f F$ can be obtained by comparison to $B \times_f S_{K_F}$. For *positive* warping function, this method was used by Chen in the CBA case with $B = \mathbf{R}$ [C1, C2], and extended to general B in [AB1, Theorem 4.1].

Note that if $f(p) = 0$, the fiber at p is collapsed to a point and identified with p. Thus for a minimizer γ in $B \times_f F$ we write $\gamma = (\gamma_B, \gamma_F)$ on the understanding that γ_F is indeterminate where $f \circ \gamma_B$ vanishes. We say γ is *horizontal* if γ_F on its determinate domain is constant. Then γ lies in an isometric copy of B, and hence γ_B is a minimizer.

Proposition 7.1. *Let B and F be CAT(K) and CAT(K_F), respectively, and $f : B \to \mathbf{R}_{\geq 0}$ be an $\mathcal{F}K$-convex function that is Lipschitz on bounded sets, where $K \leq 0$. If $X = f^{-1}(0)$ is nonempty and $K_F > 0$, suppose $f'(0^+) \geq \sqrt{K_F}$ at the footpoint of every minimizer to X.*

1. *If $X = f^{-1}(0)$ is nonempty, any minimizer in $B \times_f F$ between two points not in X, whose projections to F are $\geq \pi/\sqrt{K_F}$ apart, intersects X and consists of two horizontal segments joined by a segment in X.*
2. *If $B \times_f S_{K_F}$ is CAT(K), then so is $B \times_f F$.*

Proposition 7.2. *Let B and F be Alexandrov spaces with CBB by K and K_F respectively. Suppose $f : B \to \mathbf{R}_{\geq 0}$ is a locally Lipschitz $\mathcal{F}K$-concave function vanishing on ∂B. If ∂B is nonempty, suppose $K_F > 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in \partial B$.*

(1) *If $\partial B = f^{-1}(0)$ is nonempty, any minimizer in $B \times_f F$ joining two points not in ∂B, and intersecting $\partial B = f^{-1}(0)$, consists of two horizontal segments whose projections to F are $\pi/\sqrt{K_F}$ apart, joined by a point of ∂B.*
(2) *If $B \times_f S_{K_F}$ has CBB by K, then so does $B \times_f F$.*

For the proofs, we need to understand the behavior of minimizers when the warping function is allowed to vanish. The following lemma is an extension of Lemma 2.3 (b), according to which minimizers in a warped product (with positive warping function) are not sensitive to the fiber, only to the distance between the endpoint projections to the fiber.

Lemma 7.3. *Suppose B, F and F' are intrinsic metric spaces, and $f : B \to \mathbf{R}_{\geq 0}$ is continuous. Set $X = f^{-1}(0)$, and let $\gamma = (\gamma_B, \gamma_F)$ be a minimizer in $B \times_f F$.*

(1) *If γ has an endpoint on X, then γ_B is a minimizer in B.*
(2) *If γ intersects X, then γ is horizontal on each determinate subinterval.*
(3) *Let γ'_F map into F' with the same domain and indeterminate subdomain as γ_F, either as a minimizer with the same length and speed as γ_F if γ does not intersect X; or with constant values on the (countably many) determinate subintervals if γ intersects X, where the distance between the endpoint values is the same for γ'_F as for γ_F if both endpoints are outside of X, and the images of the other determinate subintervals are arbitrary. Then $\gamma' = (\gamma_B, \gamma'_F)$ is a minimizer in $B \times_f F'$ of the same length as γ.*

Proof. Since projection to B is nonexpanding, a horizontal minimizer from a point of X to any other point may be formed by pairing a minimizer in B with a constant map into F; if the other point is also in X, any constant value in F may be chosen. Changing the constant F-value on a determinate subinterval does not change the length, and any curve that does not project to a minimizer in B is clearly longer. This verifies (1) and (2).

Part (3). By Lemma 2.3 (a) and part (2), the construction of γ' guarantees it to have the same length as γ. If there were a shorter curve in $B \times_f F'$, it could not avoid X or else there would be a shorter curve in $B \times_f F$ with the same projection to B and a minimizing projection to F. And it could not intersect X in a point p, since then there would be a no-longer horizontal curve in $B \times_f F'$ joining p to each endpoint. Hence there would be a curve in $B \times_f F$ of the same type with the same length. Therefore γ' is a minimizer. □

Proof of Proposition 7.1.
Since $X = f^{-1}(0)$ is convex, any minimizer γ intersects X in at most an interval. Therefore γ consists of two horizontal segments joined by a segment in X.

Part (1). We may normalize so $K_F = 1$. By way of contradiction, assume there is a minimizer γ in $B \times_f F$ whose endpoints project to points of the fiber at distance $\geq \pi$ and which does not intersect X. By Lemma 7.3, there is such a minimizer also in $B \times_f \mathbf{R}$. Thus it suffices to assume $F = \mathbf{R}$.

Since γ has a subsegment whose projection to $\mathbf{R}$ has length π, there is a minimizer in $B \times_f S^1 = (B \times_f \mathbf{R})/\mathbf{Z}$ not intersecting X and whose endpoints project to opposite points of S^1. But then there are two minimizers in $B \times_f S^1$ with those endpoints, in contradiction to Theorem 6.1, which states that $B \times_f S^1$ is CAT(K). □

Part (2). We show that any triangle $\triangle = (\triangle_B, \triangle_F)$ in $B \times_f F$ is K-thin:

(i) Suppose $\triangle$ does not intersect X and satisfies $\operatorname{per} \triangle_F < \pi/\sqrt{K_F}$. Let $\triangle'_F$ be a comparison triangle in S_{K_F} for $\triangle_F$ with the same parametrization of sides. By Lemma 7.3, $\triangle' = (\triangle_B, \triangle'_F)$ is a triangle in $B \times_f S_{K_F}$ with the same sidelengths as $\triangle$. Let $\bar\gamma = (\bar\gamma_B, \bar\gamma_F)$ be a transversal minimizer for $\triangle'$. Since $\triangle'$ is K-thin by hypothesis, it suffices to construct a curve in $B \times_f F$ joining the corresponding points of $\triangle$ and no longer than $\bar\gamma$. This we do as follows. If $\bar\gamma$ does not intersect X, then $\bar\gamma_F$ is a reparametrized transversal of $\triangle'_F$, by Lemma 2.3 (a), and so is not shorter than the corresponding transversal γ_F of $\triangle_F$. In this case, pair $\bar\gamma_B$ with a reparametrization of γ_F proportional to $\bar\gamma_F$. If $\bar\gamma$ intersects X, pair $\bar\gamma_B$ with the curve that is indeterminate on the same interval as $\bar\gamma_F$ and takes as its constant values, the two points corresponding to the endpoints of $\bar\gamma_F$. Thus $\triangle$ is K-thin.

(ii) Suppose $\triangle$ has a vertex in X. Clearly $\triangle$ is K-thin if it has a side in X, since then $\triangle$ lies in an isometric copy of B. Otherwise, a vertex pair lies outside X. If the side of $\triangle$ joining this pair intersects X, then the image of $\triangle_F$ is their projections to F, and otherwise it is the segment joining their projections to F. By Lemma 7.3, we may construct a triangle $\triangle' = (\triangle_B, \triangle'_F)$ in $B \times_f \mathbf{R}$ whose sides have the same domains, indeterminate subdomains and lengths as those of $\triangle$, and where the images of $\triangle_F$ and $\triangle'_F$ correspond: either two points the same distance apart, or segments of the same length. Therefore the distances between the projections to F and $\mathbf{R}$ of any corresponding pairs of points on $\triangle$ and $\triangle'$, respectively, are equal when they are determinate. By Lemma 7.3, it follows that we can construct transversal minimizers for $\triangle'$ of the same length as the corresponding ones for $\triangle$. Since $B \times_f \mathbf{R}$ is CAT(K) by Theorem 6.1, $\triangle'$ is K-thin and hence so is $\triangle$.

(iii) If a side of $\triangle$ meets X, then $\triangle$ may be subdivided into two triangles of type (ii), which are therefore thin. Therefore $\triangle$ is K-thin, by the basic Alexandrov Lemma (see [BBI, p.115]).

It remains to deal with the cases where $\triangle$ does not meet X and $\operatorname{per} \triangle_F \geq \pi/\sqrt{K_F}$:

(iv) If a minimizer from a vertex of $\triangle$ to a point of the opposite side meets X, then $\triangle$ is subdivided into two triangles of type (iii). Since each of these is K-thin, so is $\triangle$.

(v) If the minimizers joining a vertex of $\triangle$ to the points of its opposite side α never meet X, then these minimizers have length $< \pi/\sqrt{K_F}$ by part (1). Thus by subdividing α, we may subdivide $\triangle$ into triangles of type (i). Since each of these is K-thin, so is $\triangle$. □

Proof of Proposition 7.2.

A minimizer in B either lies in ∂B or intersects ∂B only at endpoint(s) [Pm]. It follows that if γ is a minimizer in $B \times_f F$, any subsegment of γ with an endpoint in ∂B is horizontal, and either lies in ∂B or intersects it only at one or both endpoints. Therefore if γ intersects ∂B, it does so either in a single point or only at the two endpoints.

Part (1) of this theorem is the only result from §5 - 8 that will be used in the proofs of Theorems 4.2 and 4.1, and hence the only one that must be proved from first principles without reference to those theorems:

Part (1). Set $K_F = 1$, and hence $\operatorname{diam} F \leq \pi$. By Lemma 7.3, if there is a minimizer $\gamma = (\gamma_B, \gamma_F)$ consisting of two horizontal segments whose projections to F

are at distance $< \pi$, then there is a minimizer in $B \times_f [-\pi/2, \pi/2]$ with the same property. Therefore if suffices to show there are no such minimizers when $F = I$.

By hypothesis, if $p \in \partial B$, then $f(p) = 0$ and $Df_p \leq 1$. Let sn_k be the solution of $y'' + ky = 0$ such that $\mathrm{sn}_k(0) = 0, \mathrm{sn}'_k(0) = 1$. Then along any minimizer σ in B starting from ∂B we have $f(\sigma(s)) \leq \mathrm{sn}_K(s)$, an immediate consequence of the $\mathcal{F}K$-concavity of f.

Let $\gamma : [0, L] \to B \times_f I$ be a minimizer intersecting ∂B, where $q_1 = \gamma_B(0)$ and $q_2 = \gamma_B(L)$ are not in ∂B. Let $\widetilde{B}$ be the double of B and $\widetilde{q}_1, \widetilde{q}_2$ the reflections of q_1, q_2 in the other half of $\widetilde{B}$. Then a minimizer from q_1 to $\widetilde{q}_2$ will consist of two segments σ_1 and $\widetilde{\sigma}_2$ joined at some point $p \in \partial B$; the reflections $\widetilde{\sigma}_1, \sigma_2$ of $\sigma_1, \widetilde{\sigma}_2$ form together a minimizer from $\widetilde{q}_1$ to q_2. Clearly σ_1 and σ_2 make up a shortest path among those connecting q_1 and q_2 which pass through a point of ∂B. See Figure 1.

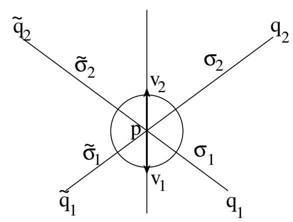

FIGURE 1

The direction space $\widetilde{\Sigma}_p$ of $\widetilde{B}$ has two pairs of directions, $q'_1, \widetilde{q}'_2$ and $q'_2, \widetilde{q}'_1$, with distance each π, so that $\widetilde{\Sigma}_p$ is a spherical suspension over each pair. Hence there are unique minimizers $q'_1{}^\frown q'_2$, $q'_2{}^\frown \widetilde{q}'_1$, $\widetilde{q}'_1{}^\frown \widetilde{q}'_2$, $\widetilde{q}'_2{}^\frown q'_1$ which together form a periodic geodesic of length 2π. This geodesic has an involutive isometry fixing its intersections with $\partial\Sigma_p$ (the direction space of ∂B), and hence intersects $\partial\Sigma_p$ in an opposite pair v_1, v_2.

An exception occurs when $q'_1 = q'_2$, whereupon p is the nearest point in ∂B to q_1 and q_2 and the longer of σ_1, σ_2 is a common minimizer $\sigma : [0, R] \to B$ through them to p. Define a map $\Psi : Y \to B \times_f I$, where Y is a sector in S_K with polar coordinates (r, θ), $0 \leq r \leq R$, $-\pi/2 \leq \theta \leq \pi/2$, and $\Psi(r, \theta) = (\sigma(r), \theta)$. Then Ψ is nonexpanding since $f(\sigma(r)) \leq \mathrm{sn}_K(r)$; but also Ψ preserves distance along radial geodesics. The image of the minimizer in Y from $(d(p, q_1), \gamma_I(0))$ to $(d(p, q_2), \gamma_I(L))$ is a path connecting $(q_1, \gamma_I(0))$ and $(q_2, \gamma_I(L))$ and has length $\leq d(p, q_1) + d(p, q_2)$, with equality only if $\ell = \pi$. This proves the exceptional case.

For the general case we define another nonexpanding map Ψ from a constant-curvature half-ball Y in S^3_K to $B \times_f I$. Represent Y as $\mathbf{D} \times_\Phi I$, where $\mathbf{D}$ is a half-disk with radius $R = \max\{d(p, q_1), d(p, q_2)\}$ and constant curvature K; when $K = 0$, take $\Phi = d_\Xi$, where Ξ is the diameter of $\mathbf{D}$; generally, take $\Phi = \mathrm{sn}_K \circ d_\Xi$. Set

$$\Psi = \exp_p \circ \Psi_{\mathbf{D}} \times \mathrm{id} : Y = \mathbf{D} \times_\Phi I \to B \times_f I.$$

Here, $\Psi_{\mathbf{D}} : \mathbf{D} \to B$ is the K-cone map into the K-cone over the geodesic in Σ_p from v_1 to v_2 through q'_1 and q'_2; this first composite is an isometric injection. (When $K = 0, -1, 1$, these K-cones are the last three on the list preceding Example 1.4 in §1.2). The map $\exp_p$ is the *gradient exponential map* defined in [PP] and proved there to be nonexpanding and isometric along cone radii which correspond to minimizers to p such as σ_1, σ_2.

We show that Ψ retains the nonexpanding property. It also retains the isometry for those horizontal minimizers from the center which are mapped to minimizers from p. Consider the functions $f \circ \exp_p \circ \Psi_{\mathbf{D}}$ and Φ on $\mathbf{D}$. For $q \in \mathbf{D}$ there is a minimizer ρ from q to the diameter of $\mathbf{D}$, and, since $\Psi_{\mathbf{D}}$ and $\exp_p$ map boundaries into boundaries, the image of ρ in B is a curve no longer than $d_{\Xi}(q)$ from $\exp_p \circ \Psi_{\mathbf{D}}(q)$ to ∂B. That is, $d_{\partial B}(\exp_p \circ \Psi_{\mathbf{D}}(q)) \leq d_{\Xi}(q)$. Since sn_K is increasing for all arguments under consideration, $f(\exp_p \circ \Psi_{\mathbf{D}}(q)) \leq \mathrm{sn}_K(d_{\partial B}(\exp_p \circ \Psi_{\mathbf{D}}(q)) \leq \Phi(q)$. So for a curve in Y, the base component is contracted by $\exp_p \circ \Psi_{\mathbf{D}}$, while the I-component is mapped identically, but is shortened more by the warping function f than it was by the warping function Φ. Thus there is an image of a geodesic in Y connecting $(q_1, \gamma_I(0)), (q_2, \gamma_I(L))$ and no longer than $d(p, q_1) + d(p, q_2)$, with equality only if $\ell = \pi$. $\square$

Part(2). Now we show that any triangle $\triangle = (\triangle_B, \triangle_F)$ in $B \times_f F$ is K-thick.

(i) Suppose $\triangle$ does not intersect ∂B, and $\mathrm{per}\, \triangle_F < 2\pi/\sqrt{K_F}$. Then we may proceed as in case (i) of Proposition 7.1 (2) to show that $\triangle$ has a comparison triangle in $B \times_f S_{K_F}$ such that the transversals of the former are no shorter than the corresponding transversals of the latter. The only change is that, rather than starting with a transversal of the model triangle, here we start with a transversal of $\triangle$ and construct a curve in $B \times_f S_{K_F}$ that is no longer. Since triangles in $B \times_f S_{K_F}$ are K-thick, so is $\triangle$.

(ii) Suppose $\triangle$ has a vertex in ∂B. If there are two such vertices, then all sides of $\triangle$ are horizontal. Thus the projection of $\triangle$ to B is a triangle with the same sidelengths. Since this triangle in B is K-thick, and the projection is nonexpanding, $\triangle$ is K-thick.

If exactly one vertex of $\triangle$ lies in ∂B, the two adjacent sides are horizontal, hence have constant projections to F. Since $\mathrm{diam}\, F \leq \pi/\sqrt{K_F}$, we may choose two points in S_{K_F} the same distance apart. Then, as in case (ii) of Proposition 7.1 (2), we may construct a comparison triangle $\triangle'$ in $B \times_f S_{K_F}$ whose transversal distances are equal to those of $\triangle$. Since $\triangle'$ is K-thick, so is $\triangle$.

(iii) Suppose $\triangle$ does not intersect ∂B and $\mathrm{per}\, \triangle_F = 2\pi/\sqrt{K_F}$, $K_F > 0$. Recall that no triangle in F can have larger perimeter [BGP]. If two vertices are $\pi/\sqrt{K_F}$ apart, F is a spherical susension with poles at these vertices (see [BBI, p.369]). Otherwise, $\triangle_F$ is spanned in F by an isometric copy of a hemisphere in S_{K_F} (see [Pl, p.836]). Thus in either case, $\triangle_F$ is spanned by an isometric copy of a sector of a hemisphere in S_{K_F}. By Lemma 7.3, $\triangle$ has a comparison triangle $\triangle'$ in $B \times_f S_{K_F}$ with the same transversal distances. Since $\triangle'$ is K-thick, so is $\triangle$.

(iv) The only remaining possibility is that no vertex of $\triangle$ lies in ∂B but some side intersects ∂B. Since $\partial B \neq \emptyset$, $K_F > 0$ by hypothesis. In this case, the projection to F of the side intersecting ∂B consists of two points at distance π, by part (1). Therefore F is a spherical suspension with poles at these points. The image of $\triangle_F$ is either the poles or two segments whose union is a minimizer of length π joining them. Let $\triangle'_F$ map into S_{K_F} with the same domains and indeterminate subdomains of the sides as $\triangle_F$, and a corresponding image. By Lemma 7.3, $\triangle' = (\triangle_B, \triangle'_F)$ is a model triangle in $B \times_f S_{K_F}$ for $\triangle$ and has the same transversal distances. Since $\triangle'$ is K-thick, so is $\triangle$. $\square$

7.2. **Reduction to 1-dimensional fiber.** By §7.1, in order to prove Theorems 1.1 and 1.2, we may assume the fiber is the model space S_{K_F}. Now let us express the model space as a warped product. As before, let cs_{K_F} denote the solution to the initial-value problem $y'' + K_F y = 0$ with $y(0) = 1, y'(0) = 0$, and I be the interval containing 0 on which $y \geq 0$. Then $S_{K_F} = I \times_{\mathrm{cs}_{K_F}} J$, where J is $\mathbf{R}$ if $K_F \leq 0$ or a circle of twice the length of I if $K_F > 0$. According to the formula for the square of the speed of a curve we can then reassociate the iterated warped product as follows:

$$B \times_f S_{K_F} = B \times_f (I \times_{\mathrm{cs}_{K_F}} J) = (B \times_f I) \times_{(f\circ\pi_B)\cdot(\mathrm{cs}_{K_F}\circ\pi_I)} J.$$

Together with Propositions 7.1 and 7.2, this expression reduces the proofs of Theorems 1.1 and 1.2 to two applications of the 1-dimensional fiber case. We complete this program in the next section.

8. Proofs of the main theorems

8.1. **Proof of Theorem 1.1 for $\mathbf{K \leq 0}$.** Suppose the warping function f is Lipschitz on bounded sets. By scaling, take $K \in \{-1, 0\}$. By Theorem 6.1(1), $B \times_f \mathbf{R}$ is CAT(K), and hence so is $B \times_f I$. By Proposition 3.1 and Theorem 4.1, the function $\Phi = (f \circ \pi_B) \cdot (\mathrm{cs}_{K_F} \circ \pi_I)$ is $\mathcal{F}K$-convex. If $K_F \leq 0$, so that $J = \mathbf{R}$, it follows immediately from Theorem 6.1(1) that $(B \times_f I) \times_\Phi J = B \times_f S_{K_F}$ is CAT(K).

On the other hand, suppose $K_F = 1$. Set $X = f^{-1}(0)$. By assumption, $\inf Df_p \leq -\sqrt{1 - Kf(p)^2}$ for any $p \notin X$. By a Zorn's Lemma argument (as in Lemma 4.2(1) of [AB2]), there is a curve from X to p along which $f' \geq 1 - \epsilon$. In particular, $X \neq \emptyset$ and $f \geq d_X$.

We wish to apply Theorem 6.1 (2) to show that $(B \times_f I) \times_\Phi J$ is CAT(K), where $I = [-\pi/2, \pi/2]$, $J = S^1$, and $\Phi(p, \theta) = f(p) \cos\theta$ for $(p, \theta) \in B \times_f I$. The zero set of Φ is Ξ, the image in $B \times_f I$ of $B \times \{-\pi/2\} \cup B \times \{\pi/2\}$; the two copies of B are identified on X, which is totally convex in B. By Theorem 6.1 (2), it suffices to show that along any minimizer γ in $B \times_f F$ from Ξ, we have $\Phi'(0^+) \geq 1$.

Suppose $\gamma_B(0) = q \in X$. In this case, γ is horizontal, say $\gamma = (\gamma_B, \theta_0)$. Set $a = (f \circ \gamma_B)'(0^+)$. Then $\Phi'(0^+) = a \cos\theta_0$. Since $f \geq d_X$, we have $a \geq 1$. Now consider $\mathbf{D} = \{\gamma_B(t)\} \times_f I \subset B \times_f I$. In its intrinsic metric, $\mathbf{D}$ is isometric to a 2-dimensional warped product. Its fiber at $\gamma_B(0) = q$ is identified with q, and its sector angle at q between $\theta = \theta_1$ and $\theta = \theta_2$ is $a|\theta_2 - \theta_1|$. The sector angle between $\theta = \theta_0$ and $\theta = \pi/2$ (or $-\pi/2$) is at least $\pi/2$, since otherwise one could replace an initial segment of γ by a curve in $\mathbf{D}$, obtaining a shorter path from Ξ to the same righthand endpoint. It is easy to check that along a Euclidean quarter-circle of radius $a \geq 1$ in the first quadrant, the arclength from $(0, a)$ to a point with $x = a\cos\theta < 1$ is less than $\pi/2$. It follows that $a \cos\theta_0 \geq 1$, hence $\Phi'(0^+) \geq 1$.

Now suppose $\gamma(0) = (q, \pi/2)$ where $q \notin X$. Then the tangent cone $C_{(q,\pi/2)}(B \times_f I)$ is the Cartesian product $C_q(B) \times \mathbf{R}_{\geq 0}$, and $\gamma'(0)$ has vanishing projection to the first factor. Since $\gamma'(0)$ is a unit vector in the warped product metric, $\Phi'(0^+) = f(q) f(q)^{-1} \frac{d}{dt}|_0 \cos(\pi/2 - t) = 1$ along γ. Thus we have shown that in every case, $(B \times_f I) \times_\Phi J = B \times_f S_{K_F}$ is CAT(K). By Proposition 7.1 (2), $B \times_f F$ is CAT(K).

Finally, we remove the Lipschitz condition on f when B is locally compact. By Lemma 2.1, f is the limit of a monotonically increasing sequence of $\mathcal{F}K$-convex functions f_i, each of which is Lipschitz on bounded sets and satisfies the first-order condition (3.1) of Proposition 3.1. Therefore, as we have just proved, $B \times_{f_i} F$

is CAT(K). By local compactness and Cohn-Vossen's theorem, closed bounded subsets of B are compact, and so the f_i converge uniformly to f on bounded sets. It follows that $d_i(P,Q) \to d(P,Q)$ for any $P,Q \in B \times F$, where d_i and d are distances in $B \times_{f_i} F$ and $B \times_f F$, respectively. Theorems from [AB1], namely Theorems 6.1 (due to Buyalo [By]) and 6.3, imply that $B \times_f F$ is CAT(K). □

8.2. **Proof of Theorem 1.2 for $\mathbf{K \leq 0}$.** The boundary condition (†) allows us to assume without loss of generality that either $\partial B = \emptyset$ and $f > 0$, or $\partial B \neq \emptyset$ and $f^{-1}(0) = \partial B$. In the remaining cases, it follows that $B^\dagger \times_{f^\dagger} F$ has CBB by K, and then it is immediate from symmetry that so does $B \times_f F$.

We wish to prove that $B \times_f F$ has CBB by K. By Proposition 3.1 and Theorem 4.2, the function $\Phi = (f \circ \pi_B) \cdot (\mathrm{cs}_{K_F} \circ \pi_I)$ is $\mathcal{F}K$-concave. By Proposition 7.2 (2), it suffices to show that $(B \times_f I) \times_\Phi J = B \times_f S_{K_F}$ has CBB by K.

Suppose $K_F \leq 0$. Then $\partial B = \emptyset$, since otherwise there would exist $q \in \partial B$ with $\sup Df_q(v) > 0$, and hence by hypothesis K_F would be positive. Since $I = J = \mathbf{R}$, $\Phi > 0$, and $\partial(B \times_f I) = \emptyset$, it is immediate from Theorem 6.2(1) that both $B \times_f I$ and $(B \times_f I) \times_\Phi J$ have CBB by K.

Suppose $K_F = 1$. Then $I = [-\pi/2, \pi/2]$. If $\partial B = \emptyset$ and $f > 0$, then $B \times_f I$ has CBB by K by Theorem 6.2(1). If $\partial B \neq \emptyset$, so f vanishes on ∂B, then $Df \leq K_F = 1$ on ∂B by hypothesis. Then $B \times_f I$ has CBB by K by Theorem 6.2(2). Since $K_F = 1$, we have $J = S^1$ and $\Phi(p,\theta) = f(p)\cos\theta$. The zero set of Φ is Ξ, the image in $B \times_f I$ of $B \times \{-\pi/2\} \cup B \times \{\pi/2\}$; the two copies of B are identified on ∂B. Since $\partial(B \times_f I) = \Xi$, the boundary condition (†) for Φ is trivially satisfied. By Theorem 6.2(2), it only remains to verify that $\Phi'(0^+) \leq 1$ along any geodesic γ starting from Ξ. If $\gamma(0) = (q,\theta)$ where $q \in \partial B$, then γ is horizontal, say $\gamma = (\gamma_B, \theta_0)$. Then $(\Phi \circ \gamma)'(0^+) = (\cos\theta_0)(f \circ \gamma_B)'(0^+) \leq 1$, since $(f \circ \gamma_B)'(0^+) \leq 1$ by hypothesis. On the other hand, if $\gamma(0) = (q, \pm\pi/2)$ where $q \notin X$, then the desired inequality holds for the same reason as in the preceding proof. □

8.3. **Proofs of Theorems 1.1 and 1.2 for $\mathbf{K > 0}$.** As in the proof of Theorem 3.1, case ($K > 0$), if f satisfies conditions (1) and (2) of our theorems for $K = 1$, then its linear homogeneous extension $C(f)$ satisfies the same conditions for $K = 0$. Then Theorems 1.1 and 1.2 hold for $K = 1$ because they hold for $K = 0$. Indeed, the $K = 1$ hypotheses on f imply that $C(B) \times_{C(f)} F$ has 0 as a curvature bound. Since

$$C(B) \times_{C(f)} F = (\mathbf{R}_{\geq 0} \times_r B) \times_{C(f)} F = \mathbf{R}_{\geq 0} \times_r (B \times_f F) = C(B \times_f F),$$

then $B \times_f F$ has 1 as a curvature bound. □

Remark 8.1. The case $K < 0$ cannot be obtained in a similar way from spaces with curvature bounds of $K = 0$ in the energy sense ([AB3]; compare proof of Theorem 6.2). Indeed, that would require an extensive theory, including a version of our main theorems, for such spaces.

9. First variation for billiard trajectories

It only remains to prove Theorems 4.1 and 4.2, on the lifting of convex functions to a warped product. Our method is to approximate warped product geodesics by strip space geodesics. For this, we use first variation inequalities (§9) and approximate energy equations (§10) for strip space geodesics. In §9 and §10, we work locally in a region where $f > 0$.

Consider a billiard trajectory in a half-strip given by inequalities $0 \le u \le \epsilon f$ in $B_{\mathrm{cs}_k} \times \mathbf{R}$, $k \le 0$ (see Figure 2). Here, a *billiard trajectory* is a curve whose unfolding into the strip space W_ϵ is a minimizer. We need inequalities equivalent to the equality of the angles of incidence and reflection in the smooth case.

Assume B is a geodesic metric space with either an upper or lower curvature bound. Then the first variation of segment length is given by the usual formula, assuming sufficiently short segments (CBA) or existence of geodesic extensions (CBB), hence in all of our applications. Assume $f : B \to \mathbf{R}$ is positive, Lipschitz and either $\mathcal{F}K$-convex or $\mathcal{F}K$-concave.

Our analysis is localized to a billiard segment β that starts at a midlevel point $(m_-, 0)$ of a strip, has a left part of length h_- ending at the reflection point $(p, \epsilon f(p))$, and continues from there on a right part of length h_+ to a midlevel point $(m_+, 0)$ (see Figure 2). Thus β represents a minimizer in W_ϵ connecting the midlevels of adjacent strips. β is varied with endpoints fixed, by moving p along some geodesic σ and consequently moving the reflection point along the graph $u = \epsilon f$ above σ.

Arclength parameters on the two parts are denoted t_-, t_+, having value 0 at $(p, \epsilon f(p))$ and increasing from left to right; the projections to B are geodesic segments $\sigma_\pm$, with lengths $\ell_\pm$, and forming an angle τ at p. The horizontal and vertical components of the speeds of $\sigma_\pm$ are denoted by $a_\pm, b_\pm$; when $k = 0$, they are constants $a_\pm = \ell_\pm / h_\pm$ and $b_\pm = \epsilon f(p)/h_\pm$. For $k < 0$, $a_\pm, b_\pm$ are functions of the arclength $t_\pm$, but we use the same notation in (9.1) to denote their values at $t_\pm = 0$.

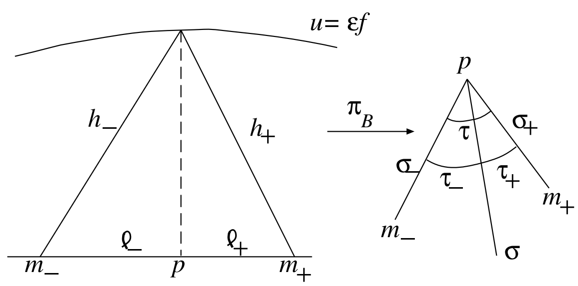

FIGURE 2

Theorem 9.1. *Let the billiard segment β vary with endpoints fixed, according to a variation of p along a geodesic segment σ in B. If σ makes angles $\tau_\pm$ with $\sigma_\pm$, then, setting $f' = Df(\sigma'(0))$,*

$$\epsilon(b_- + b_+) f' \ge (a_- \cos\tau_- + a_+ \cos\tau_+)\, \mathrm{cs}_k(\epsilon f(p)). \tag{9.1}$$

Proof. Suppose $k = 0$ (so $\mathrm{cs}_k = 1$). The quantities $h_\pm = \sqrt{\ell_\pm^2 + \epsilon^2 f^2}, \ell_\pm$ are functions of the arclength s of σ, where we have denoted the restriction of f along σ simply by f. The one-sided derivatives with respect to s at $s = 0$ will be denoted with a prime. Using the formula for first variation in B we get

$$h'_\pm = \frac{-\ell_\pm \cos\tau_\pm + \epsilon^2 f(p) f'}{h_\pm}.$$

The sum $h_- + h_+$ attains a minimum at $s = 0$, so that adding the two derivatives gives a nonnegative value; expressing the result in terms of $a_\pm \geq 0, b_\pm \geq 0$ and manipulating the inequality gives our fundamental result.

When $k < 0$ the idea is the same, but the part of $B_{\mathrm{cs}_k} \times \mathbf{R}$ projecting to a geodesic segment in B is isometric to a strip in the hyperbolic plane, so hyperbolic trigonometry must be used. For later reference we note the resulting formula for the vertical speeds:

$$b_\pm = \frac{\sinh(\sqrt{k}\epsilon f(p))\cosh(\sqrt{k}(h - t_\pm))}{\sqrt{\sinh^2(\sqrt{k}h) + \sinh^2(\sqrt{k}\epsilon f(p))\sinh^2(\sqrt{k}(h - t_\pm))}}. \tag{9.2}$$

□

Remark 9.2. Conversely, a curve β in W_ϵ, connecting the midlevels of two adjacent strips and consisting of two geodesic segments joined at a point $(p, \epsilon f(p))$, is a billiard segment if it satisfies (9.1) for all endpoint-fixed variations. In the CBB case, the inequalities (9.1) become equations (corresponding to the splitting of the direction cone of the strip space W_ϵ, at a point where a geodesic of W_ϵ passes through a gluing seam).

Corollary 9.3. *There is always a direction v at p such that, setting $f' = Df(v)$,*

$$\epsilon(b_- + b_+)f' \geq |a_- - a_+|.$$

Proof. Suppose $\tau < \pi$. In the Euclidean triangle with two sides a_-, a_+ and included angle τ, let the median within that included angle form angles τ_-, τ_+ with the respective sides. We can choose the direction v so that it makes the same angles $\tau_\pm$ with $\sigma_\pm$. Then it can be calculated that twice the length of the median is

$$a_- \cos\tau_- + a_+ \cos\tau_+ = \sqrt{(a_- - a_+)^2 + 2a_-a_+(1 + \cos\tau)} \geq |a_- - a_+|,$$

so the result follows from (9.1). For the case $\tau = \pi$ take (τ_-, τ_+) to be either $(0, \tau)$ or $(\tau, 0)$ so as to make the right side of (9.1) equal to $|a_- - a_+|\,\mathrm{cs}_k(\epsilon f(p))$, showing that one of the directions of $\sigma_\pm$ can be used for v. □

10. The energy equation for strip approximations

Here we show that strip space geodesics satisfy an approximate energy equation.

As in §9, assume B is a geodesic metric space with either an upper or lower curvature bound, and $f > 0$ is Lipschitz and either $\mathcal{F}K$-convex or $\mathcal{F}K$-concave. Consider a nonhorizontal geodesic $\gamma : [0, L] \to B \times_f \mathbf{R}$, and an approximating sequence of stripspace geodesics $\gamma_n : [0, L_n] \to W_\epsilon$, which converges to γ via homeomorphisms $\psi_\epsilon : W_\epsilon \to B \times_f \mathbf{R}$. Here, the restriction of ψ_ϵ to the ith strip $W_\epsilon^{(i)}$ is the inverse of the map $\varphi_\epsilon^{(i)}$ of (5.2). We assume $L_n \to L$ and $f_n|[0, T]$ converges uniformly to $f|[0, T]$ for $0 < T < L$. We specify γ_n and the relation between n and ϵ by requiring γ_n to consist of a chain of product segments γ_{ni} in $W_\epsilon^{(i)}$, $i = 0, \ldots, n$. Assume γ_n starts and ends in the midlevels of $W_\epsilon^{(0)}$ and $W_\epsilon^{(n)}$, at points $\varphi_\epsilon^{(0)}(\gamma(0))$ and $\varphi_\epsilon^{(n)}(\gamma(L))$ respectively. If $[0, H]$ is the projection of γ to $\mathbf{R}$, then $\epsilon = H/2n$.

We shall write $f_n = f \circ \pi_B \circ \gamma_n$ and $f = f \circ \pi_B \circ \gamma$. By Lemma 2.3 (c), the projection $\gamma_{\mathbf{R}} = \pi_{\mathbf{R}} \circ \gamma$ satisfies $\gamma_{\mathbf{R}}' = c_\gamma / f^2$, where c_γ is a constant.

Proposition 10.1. *The vertical speed b_n of the strip space geodesics converges uniformly to c_γ / f.*

Remark 10.2. Say a function $g : X \to Y$ from one metric space to another is quasi-Lipschitz with *window constant* δ if $d(g(x_1), g(x_2)) \leq Cd(x_1, x_2) + \delta$ for every $x_1, x_2 \in X$. For example, if we approximate a Lipschitz continuous $g : \mathbf{R} \to \mathbf{R}$ by a step function with intervals of constancy no longer than $\delta/2C$, then the approximation is quasi-Lipschitz with window constant δ. If a sequence $g_n : [0,T] \to \mathbf{R}$ converges pointwise to g and satisfies quasi-Lipschitz conditions with window constants $\delta_n \to 0$ and uniform Lipschitz constant C, then g is Lipschitz with constant C and the convergence is uniform.

Proof of Proposition 10.1. By assumption, on a bounded neighborhood U of $\pi_B \circ \gamma$, there is a uniform bound $|Df| \leq C$. It follows that f_n converges uniformly to f. Moreover, f is bounded on U: $0 < A < f < A'$. We deal only with n sufficiently large that $\pi_B \circ \gamma_n$ lies in U.

From (5.2), the formula for ψ_ϵ on $W_\epsilon^{(i)}$ is $\psi_\epsilon(p,u) = (p, \frac{u}{f(p)} + 2i\epsilon)$. The vertical component of the speed of $\psi_\epsilon(\gamma_n)$ is $(\pi_{\mathbf{R}} \circ \gamma_n)'/f_n) - (\pi_{\mathbf{R}} \circ \gamma_n/f_n^2)f_n' = (b_n/f_n) - (\pi_{\mathbf{R}} \circ \gamma_n/f_n^2)f_n'$, which we wish to show converges uniformly to c_γ/f^2. Since we have $|\pi_{\mathbf{R}} \circ \gamma_{ni}| \leq \epsilon f_{ni}$, the second term converges uniformly to 0 and can be discarded in further calculations. Moreover, since the f_n converge uniformly to f, we can also replace them by f. By integrating these vertical components of speed we get the vertical displacement. Since the components of the transformed strip-space geodesic converge in length to those of γ we have immediately an integral form of the desired limit:

$$\lim_{n\to\infty} \int_0^T \frac{b_n}{f} dt = \int_0^T \frac{c_\gamma}{f^2} dt, \tag{10.1}$$

for all $T \in [0, L)$.

We revert to the notation of §9, denoting the left and right limit values of the piecewise analytic function b_n at a discontinuity by b_- and b_+, and similarly for the corresponding speeds a_-, a_+ of the projection of γ_n to B. Corollary 9.3 and our assumption that derivatives of f are bounded by C give: $\epsilon(b_- + b_+)C \geq |a_- - a_+|$. If $a_- \neq a_+$, we may multiply both sides of this inequality by $(a_- + a_+)/(b_- + b_+)$, which is equal to $|b_- - b_+|/|a_- - a_+|$ since $a_-^2 - a_+^2 = b_+^2 - b_-^2$, to get:

$$\epsilon(a_- + a_+)C \geq |b_- - b_+|. \tag{10.2}$$

This holds trivially when $a_- = a_+$. Furthermore, we shall only use a weaker form in which we replace $a_- + a_+$ by 2.

We know from the inequality (10.2) that the piecewise analytic functions b_n have jumps of magnitude at most $2\epsilon C$ and we also have the lower bound $2A\epsilon$ on segments of analyticity, because each segment has to cross an entire strip. On these segments of analyticity either b_n is constant ($k = 0$) or the formulas from hyperbolic trigonometry ($k < 0$) show that their derivatives with respect to t have a uniform bound A'' dependent only on the length of the longest segment. The number of jumps in an interval $[t, u]$ cannot exceed $(|t - u|/2A\epsilon) + 1$, so we have a bound $|b_n(t) - b_n(u)| \leq A''|t - u| + 2\epsilon C(|t - u|/2A\epsilon) = (A'' + C/A)|t - u| + 2\epsilon C$. This shows that b_n satisfies a quasi-Lipschitz condition with window constant $\delta_n = 2C\epsilon$ and Lipschitz constant $A'' + C/A$. It is easily shown that this is enough regularity of the integrand so that convergence of the indefinite integrals (10.1) implies pointwise convergence of the integrands to the integrand of the limit. By Remark 10.2, the convergence of integrands is uniform. □

Corollary 10.3. *The segment lengths of γ_n are bounded by $O(\epsilon)$. The differences between the horizontal and vertical speeds a_n, b_n within a segment and their segment-end values which appear in the basic inequalities (9.1) are $O(\epsilon^2)$.*

Proof. Segments of γ_n consist of two pieces separated by a strip midlevel, with lengths $h_\pm$. Geometrically each $h_\pm$ is the hypotenuse of a right triangle, as pictured in Figure 2, with a vertical leg $\epsilon f(p)$, in a plane of constant curvature $k \leq 0$. The angle opposite that leg has sine equal to the midlevel value of b_n, which we now know has a uniform lower bound c_γ/A'. Consequently, the hypotenuse inherits a bound $O(\epsilon)$ from the leg. As for the bound on the nonconstancy of the speeds, it is 0 for $k = 0$, and for $k < 0$ it follows from the Taylor expansion of (9.2). □

11. Proofs of Theorems 4.1 and 4.2

Now we use the first variation inequalities and approximate energy equations to prove Theorems 4.1 and 4.2, about convexity of the lift of f from B to $B \times_f I$.

11.1. **Integral inequalities when $f > 0$.** The following integral inequalities are the key remaining step. As specified in §10, we again consider a nonhorizontal geodesic $\gamma : [0, L] \to B \times_f \mathbf{R}$ in a region where $f > 0$, and a sequence of geodesics $\gamma_n : [0, L_n] \to W_\epsilon$, which converge to γ via the homeomorphisms $\psi_\epsilon : W_\epsilon \to B \times_f \mathbf{R}$. We write $f_n = f \circ \pi_B \circ \gamma_n$ and $f = f \circ \pi_B \circ \gamma$. Propositions 11.1 and 11.2 below give estimates on these functions.

Proposition 11.1. *Let B be a complete, finite-dimensional Alexandrov space with CBB by $K \leq 0$, $f : B \to \mathbf{R}_{\geq 0}$ be $\mathcal{F}K$-concave, and K_F be a constant. Suppose $K_F - K^2 \geq 0$ and $Df_p \leq \sqrt{K_F - Kf(p)^2}$. Then on any subinterval $[t_1, t_2]$ of $[0, L)$ with length at least $\sqrt{\epsilon}$, the change in the derivative of f_n satisfies, for some null sequence of constants ω_n:*

$$\Delta_{[t_1,t_2]} f_n' \leq \int_{t_1}^{t_2} \left[K_F \frac{c_\gamma^2}{f(t)^3} - Kf(t) + \omega_n \right] dt. \tag{11.1}$$

Proof. We prove such an integral inequality for $[-h_-, h_+]$ parametrizing a single reflection path as pictured in Figure 2 with the constants ω_n depending only on n and bounds on f. Then for longer intervals $[t_1, t_2]$ we can sum over all the complete single reflection paths parametrized within that interval. The remaining two end segments will have lengths of order $O(\epsilon)$ by Corollary 10.3; the estimate of the change in f_n' for a single reflection path will also be applicable to the end segments, showing that those two terms are also of order $O(\epsilon)$. Thus, the discrepancy due to the two end segments has a bound of the form of the integral of $O(\sqrt{\epsilon})$ on $[t_1, t_2]$, and so can be included in ω_n.

The change in f_n' for $[-h_-, h_+]$ will be decomposed into the changes on the half-open intervals $[-h_-, 0)$ and $(0, h_+]$ plus the jump at $t = 0$. (For convenience, we shift the arclength parameter to vanish at the reflection point.) Our formulas used in proving this proposition and for Proposition 11.1 are only valid for $k = 0$, but by Corollary 10.3 the discrepancy is $O(\epsilon^2)$, and so can be absorbed easily in the error term ω_n.

Denote the jump at $t = 0$ by $\Delta_0 f_n'$. Let $Df(v)$ be a maximum value of Df on the direction space at p. By Corollary 9.3, $Df(v) \geq 0$. By the 1-concavity of Df,

$$f_+' \leq Df_p(v) \cos \tau_+ \quad \text{and} \quad -f_-' \leq Df_p(v) \cos \tau_-. \tag{11.2}$$

Then

$$\begin{aligned}
\Delta_0 f_n' &= -a_- f_-' + a_+ f_+' && (11.3)\\
&\le a_- Df(v)\cos\tau_- + a_+ Df(v)\cos\tau_+ && (11.4)\\
&\le \epsilon(b_- + b_+) Df(v)^2 && (11.5)\\
&\le \epsilon(b_- + b_+)(K_F - Kf(p)^2). && (11.6)
\end{aligned}$$

Here, (11.5) is (9.1) multiplied by $Df(v) \geq 0$, and (11.6) is the first order constraint of our hypothesis.

Now substitute $\epsilon = b_\pm h_\pm / f(p)$ and write the result as an integral with constant integrand on left and right subsegments:

$$\Delta_0 f_n' \leq \int_{-h_-}^{h_+} \frac{b_n(t)^2}{f(p)}(K_F - Kf(p)^2)\,dt. \tag{11.7}$$

Here $b_n(t)$ is the vertical component of speed, regarded as a function of t.

The change in f_n' on $[-h_-, 0)$ is estimated by using the $\mathcal{F}K$-concavity of f on the projected geodesic, accounting for the change in parametrization from arclength s in B to t by the horizontal speed a_-, once for changing to derivative with respect to t and a second time for changing to integration with respect to t:

$$\Delta_{[-h_-,0)} f_n' \leq \int_{-h_-}^{0} -Ka_-^2 f_n(t)\,dt. \tag{11.8}$$

Adding (11.7) and (11.8) and the similar inequality for the right interval gives an estimate for the whole interval:

$$\Delta_{[-h_-,h_+]} f_n' \leq \int_{-h_-}^{h_+} \left(\frac{K_F b_n(t)^2}{f(p)} - K[b_n(t)^2 f(p) + a_n(t)^2 f_n(t)] \right) dt. \tag{11.9}$$

Now take $t \in [t_1, t_2]$ and p to be a piecewise-constant function of t depending on n. Summing integrals (11.9) along with the two end terms gives an integral bound for $\Delta_{[t_1,t_2]} f_n'$ which has the same form as (11.9), except for an additional term $O(\sqrt{\epsilon})$ to account for the end terms. It remains to remark that f is Lipschitz on bounded sets, since f is locally Lipschitz (Remark 3.2) and closed bounded subsets of B are compact. Thus the factors $f(p)$ and $f_n(t)$ in the integral bound for $\Delta_{[t_1,t_2]} f_n'$ converge uniformly to $f(t)$ (the former, by Remark 10.2), and $b_n(t)$ converges uniformly to $c_\gamma / f(t)$ (Proposition 10.1). Thus we can substitute the limit functions at the expense of an error term ω_n having limit 0, and finally replace $b_n(t)^2 + a_n(t)^2$ by 1. □

For the CBA case, we must reverse the sense of the inequalities. Since the basic first-variation inequality (9.1) has a fixed sense, it must appear with a nonpositive multiplier. This is one way to understand how the differing hypotheses on Df work in Proposition 3.1.

Proposition 11.2. *Let B be a CAT(K) space for $K \leq 0$, $f : B \to \mathbf{R}_{\geq 0}$ be a $\mathcal{F}K$-convex function that is Lipschitz on bounded sets, and K_F be a constant. For any $p \in B - f^{-1}(0)$ such that $K_F - Kf(p)^2 > 0$, suppose $\inf Df_p \leq -\sqrt{K_F - Kf(p)^2}$. Then on any subinterval $[t_1, t_2]$ of $[0, L)$ with length at least $\sqrt{\epsilon}$, the change in the derivative of f_n satisfies, for some null sequence of constants ω_n:*

$$\Delta_{[t_1,t_2]} f_n' \geq \int_{t_1}^{t_2} \left[K_F \frac{c_\gamma^2}{f(t)^3} - Kf(t) + \omega_n \right] dt. \tag{11.10}$$

Proof. Let $Df_p(v)$ be the minimum for Df_p. Suppose $Df_p(v) \leq 0$. Take the direction for the meaning of f' in (9.1) to be v. By the $\mathcal{F}1$-convexity of Df_p on Σ_p, (11.2) holds with the directions of the inequalities reversed. Starting with (11.3), it follows that now (11.4) holds with the direction of inequality reversed; as does (11.5) because now $Df_p(v) \leq 0$; as does (11.6), by hypothesis if $K_F - Kf(p)^2 > 0$, and trivially otherwise. On the other hand, suppose $Df_p(v) > 0$. Then by hypothesis $K_F - Kf(p)^2 \leq 0$. Hence

$$\Delta_0 f_n' \geq (a_+ + a_-)Df_p(v) \geq 0 \geq \epsilon(b_+ + b_-)(K_F - Kf(p)^2).$$

Hence in either case, $\Delta_0 f_n' \geq \epsilon(b + +b_-)(K_F - Kf(p)^2)$. The remainder of the proof is fully analogous to that of Proposition 11.1. □

11.2. **Estimates on second derivatives of smoothed warping functions.** In order to connect the integral inequalities (11.1), (11.10) to the smooth differential inequalities on the second derivative, we employ the standard mollifier technique, i.e., convolution with kernels φ_λ. We assume they satisfy: φ_λ is C^∞, nonnegative, even, has support $[-\lambda, \lambda]$, and unit mass: $\int \varphi_\lambda(t)\,dt = 1$. For a function $g : \mathbf{R} \to \mathbf{R}$ we denote the smoothed function by $M_\lambda g$, so it is given by

$$(M_\lambda g)(t) = \int g(u)\varphi_\lambda(t-u)\,du = \int g(t-u)\varphi_\lambda(u)\,du.$$

If g is absolutely continuous, the operation commutes with derivation: $(M_\lambda g)' = M_\lambda(g')$. Moreover, using integration by parts, $M_\lambda g'(t) = \int g(u)\varphi_\lambda'(t-u)\,du$. If g is uniformly continuous, then $M_\lambda g \to g$ uniformly as $\lambda \to 0$.

By smoothing the functions f, f_n we can turn the estimates of Propositions 11.1 and 11.2 into estimates on the second derivatives:

Proposition 11.3. *Under the hypothesis of Proposition 11.1,*

$$M_\lambda f'' \leq K_F c_\gamma^2 M_\lambda(\frac{1}{f^3}) - KM_\lambda f. \tag{11.11}$$

Under the hypothesis of Proposition 11.2, the reverse inequality holds.

Proof. Let $\eta = \sqrt{\epsilon}$. For a subinterval $[t, t+\eta]$ we have

$$\begin{aligned} M_\lambda f_n'(t+\eta) - M_\lambda f_n'(t) &= \int f_n'(u)\varphi_\lambda(t+\eta-u)\,du - \int f_n'(u)\varphi_\lambda(t-u)\,du \\ &= \int (f_n'(u+\eta) - f_n'(u))\,\varphi_\lambda(t-u)\,du \\ &\leq \int\int_u^{u+\eta} \left(\frac{K_F c_\gamma^2}{f(v)^3} - Kf(v) + \omega_n\right) dv\,\varphi_\lambda(t-u)\,du. \end{aligned}$$

By the mean value theorem
(11.12)

$$M_\lambda f_n'(t+\eta) - M_\lambda f_n'(t)\eta \leq \int \left(\frac{K_F c_\gamma^2}{f(v(u))^3} - Kf(v(u)) + \omega_n\right) \varphi_\lambda(t-u)\,du,$$

where $u < v(u) < u + \eta$. As $\epsilon \to 0$ the right side of (11.12) obviously goes to the right side of (11.11). To show the same for the left sides we can use a Taylor expansion:

$$\begin{aligned} M_\lambda f_n'(t+\eta) - M_\lambda f_n'(t) &= \eta M_\lambda f_n''(t) + 12\eta^2 M_\lambda f_n'''(\bar t) \\ &= \eta \int f_n(u)\varphi_\lambda''(t-u)\,du + 12\eta^2 \int f_n(u)\varphi_\lambda'''(\bar t - u)\,du, \end{aligned}$$

where $t < \bar t < t+\eta$. Dividing by η and taking the limit as $\epsilon \to 0$ does indeed give $M_\lambda f''(t)$, due to the uniform convergence $f_n \to f$.

Similarly, (11.10) becomes the same second order differential inequality with the direction reversed. □

11.3. **$\mathcal{F}K$-convexity/concavity of $f\cdot \mathrm{cs}_{K_F}$.** Now we are ready to prove Theorems 4.1 and 4.2, which we restate as:

Proposition 11.4. *Along a $B\times_f I$-geodesic, $\Phi = (f\circ\pi_B)\cdot(\mathrm{cs}_{K_F}\circ\pi_I)$ is $\mathcal{F}K$-concave under the hypotheses of Theorem 4.2, and $\mathcal{F}K$-convex under the hypotheses of Theorem 4.1.*

Proof. (i) Along a nonhorizontal geodesic γ in a region where $f>0$, we apply Proposition 11.3. Define $\theta(t,\lambda) = \pi_I(\gamma(0)) + \int_0^t c_\gamma (M_\lambda f(u))^2\,du$. In the following we abbreviate cs_{K_F} as cs. Then an easy calculation shows that in the second derivative of $M_\lambda f\cdot \mathrm{cs}(\theta)$ the terms containing $M_\lambda f'$ cancel, so that

$$\begin{aligned} &\frac{d^2}{dt^2}(M_\lambda f(t)\,\mathrm{cs}(\theta(t,\lambda))) = (M_\lambda f)''(t)\,\mathrm{cs}(\theta(t,\lambda)) - K_F\frac{c_\lambda^2}{(M_\lambda f(t))^3}\,\mathrm{cs}(\theta(t,\lambda)) \\ &\le -KM_\lambda f(t)\,\mathrm{cs}(\theta(t,\lambda)) + K_F c_\gamma^2\left((M_\lambda\frac{1}{f^3}(t) - \frac{1}{(M_\lambda f(t))^3}\right)\mathrm{cs}(\theta(t,\lambda)). \end{aligned}$$

Since $M_\lambda \frac{1}{f^3}(t) - \frac{1}{(M_\lambda f(t))^3}$ converges uniformly to 0 as $\lambda\to 0$, $M_\lambda f\cdot\mathrm{cs}(\theta)$ is $\mathcal{F}K'$-concave for $K' < K$ arbitrarily close to K as $\lambda \to 0$. But $f\cdot \mathrm{cs}$ is the uniform limit of $M_\lambda f\cdot \mathrm{cs}(\theta)$, so it is also $\mathcal{F}K'$-concave for all such K', and hence $f\cdot\mathrm{cs}$ is $\mathcal{F}K$-concave.

For the proof of the $\mathcal{F}K$-convex case we simply reverse the inequalities and let K' approach K from above.

(ii) Along horizontal geodesics, the desired property is inherited from that of f on B, since the factor cs_{K_F} is constant and the arclength is the same as on the projection to B.

(iii) The remaining cases involve points at which $f=0$:

(CBA) In this case a minimizer through a point at which $f=0$ consists of two horizontal segments separated by a (possibly trivial) segment on which $f=0$. On all three segments $f\cdot\mathrm{cs}_{K_F}$ is $\mathcal{F}K$-convex and it has a positive jump in its derivative at the connecting points, so it is obviously $\mathcal{F}K$-convex on the whole.

(CBB) If f vanishes, it does so only at boundary points. Moreover, the inequality $Df_p(v) \le \sqrt{K_F - Kf(p)^2}$ imples that $K_F>0$ if f has a vanishing point. Then $I = [-\pi/2\sqrt{K_F}, \pi/2\sqrt{K_F}]$. By Proposition 7.2 (1), a minimizer γ that intersects $f^{-1}(0)$ consists of two horizontal segments whose projections to I are $\pi/\sqrt{K_F}$ apart, joined by a point $p\in\partial B$ for which $f(p)=0$. (Recall that, in anticipation of using it here, we proved Proposition 7.2 (1) without reference to the $\mathcal{F}K$-concavity of $f\cdot\mathrm{cs}_{K_F}$.) Since the two fiber projections are $-\pi/2\sqrt{K_F}$ and $\pi/2\sqrt{K_F}$, then Φ vanishes identically along γ because $\mathrm{cs}_{K_F}\circ\pi_I$ vanishes everywhere except p. □

Acknowledgments

We are grateful to Valeri Berestovskii and Conrad Plaut for pointing out the reference [AG], and to Anton Petrunin for conversations about CBB spaces.

References

[AB1] S. B. Alexander, R. L. Bishop, *Warped products of Hadamard spaces*, Manuscripta Math. **96** (1998), 487-505.

[AB2] S. B. Alexander, R. L. Bishop, *$\mathcal{F}K$-convex functions on metric spaces*, Manuscripta Math. **110** (2003), 115-133.

[AB3] S. B. Alexander, R. L. Bishop, manuscript in preparation.

[ABN] A.D. Alexandrov, V. N. Berestovskii, I. G. Nikolaev, *Generalized Riemannian spaces*, Russian Math. Surveys **41:3** (1986), 1-54.

[AG] F. Ancel, C. Guilbault, *Interiors of compact contractible N-manifolds are hyperbolic* ($N \geq 5$), J. Differential Geometry **45** (1997), 1-32.

[Ba] W. Ballmann, *Lectures on Spaces of Nonpositive Curvature.* Basel, Boston: Birkhauser-Verlag, 1995; DMV Seminar Bd. 25.

[Be] V. Berestovskii, *Borsuk's problem on metrization of a polyhedron*, Dokl. Akad. Nauk. SSSR **27:1** (1983), 56-59.

[BN] V. N. Berestovskii, I. G. Nikolaev, *Multidimensional generalized Riemannian spaces*, in *Geometry IV. Non-regular Riemannian Geometry. Encyclopaedia of Math. Sciences* Springer-Verlag, Berlin, Heidelberg, 1993, 165-244.

[BO] R. L. Bishop, B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc.**145** (1969), 1-49.

[BH] M. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer-Verlag, 1999.

[BBI] D. Burago, Yu. Burago, S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Mathematics, Vol. 33, Amer. Math. Soc., Providence, RI, 2001.

[BFK] D. Burago, S. Ferleger, A. Kononenko, *Uniform estimates on the number of collisions in semi-dispersing billiards*, Annals of Math.(2) **147** (1998), 695-708.

[BGP] Yu. D. Burago, M. Gromov, G. Perelman, *A. D. Alexandrov spaces with curvature bounded below*, Russian Math. Surveys **47** (1992), 1-58.

[By] S. Buyalo, *Lecture notes on spaces of nonpositive curvature*, Course taught at UIUC, Spring,1995 (preprint available on request).

[CC] J. Cheeger, T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Annals of Math. (2) **144** (1996), 189-237.

[C1] C.-H. Chen, *Warped products of metric spaces of curvature bounded from above*, Thesis, UIUC, 1996.

[C2] C.-H. Chen, *Warped products of metric spaces of curvature bounded from above*, Trans. Amer. Math. Soc. **351**(1999), 4727-4740.

[G1] M. Gromov, *Hyperbolic manifolds, groups and actions.* I. Kra, B. Maskit (eds.), *Riemann Surfaces and Related Topics*, Annals of Math. Studies 97, Princeton University (1981), 183-213.

[G2] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Progress in Mathematics **152**, Birkhäuser, Basel, Boston, 1999.

[G3] M. Gromov, *CAT(κ)-spaces: construction and concentration* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov **280**(2001), Geom. i Topol. 7, 100-140, 299-300.

[HLS] C. Hummel, U. Lang, V. Schroeder, *Convex hulls in singular spaces of negative curvature* Ann. Global. Anal. Geom. **18**(2000), 191-204.

[J] J. Jost, *Nonpositive Curvature: Geometric and Analytic Aspects*, Birkhäuser, Basel, Boston, 1997.

[K] B. Kleiner, *The local structure of length spaces of curvature bounded above*, Math. Z. (1999), 409-456.

[Pm] G. Perelman, *Alexandrov's spaces with curvature bounded from below II*, preprint (1991).

[PP] G. Perelman, A. Petrunin, *Quasigeodesics and gradient curves in Alexandrov spaces*, preprint (1994), www.math.psu.edu/petrunin.

[Ps] P. Petersen, *Riemannian Geometry*, Springer, New York, 1998.

[Pn] A. Petrunin, *Applications of quasigeodesics and gradient curves.* K. Grove, P. Petersen (eds.), *Comparison geometry (Berkeley, CA, 1993–94)*, Math. Sci. Res. Inst. Publ. **30**, Cambridge Univ. Press, Cambridge (1997), 203-219.

[Pl] C. Plaut, *Metric spaces of curvature* $\geq k$. R. Daverman, R. Sher (eds.), *Handbook of Geometric Topology*, Elsevier, (2002), 819-898.

[R] Yu. G. Reshetnyak, *On the theory of spaces of curvature not greater than* K, Mat. Sbornik **52** (1960), 789-798.

1409 W. Green St., Urbana, Illinois 61801
E-mail address: sba@math.uiuc.edu

1409 W. Green St., Urbana, Illinois 61801
E-mail address: bishop@math.uiuc.edu