Chapter 1

Definitions of curvature bounded above

We begin by defining CAT$[\kappa]$ spaces and spaces of curvature $\leqslant \kappa$ via (2+2)-point comparison (Section **??**). Section **??** gives definitions in terms of angle and distance comparisons for triangles. Useful definitions in terms of convexity of distance functions and developments are formulated in Section **??**. A definition using the Kirszbraun short-map extension property may be found in Section **??**. In Chapter **??** we give another four-point definition and relate it to the (1+3)-point definition of curvature bounded below (Section **??**) and to Wald's original curvature condition.

Section **??** looks at thin triangles and their inheritance lemma.

The major globalization theorems are treated in Sections **??**, **??** and **??**: Alexandrov's patchwork theorem, and the no-conjugate-point, Hadamard–Cartan and lifting theorems.

Reshetnyak's majorization theorem, a strong generalization of the thin-triangles definition of CAT$[\kappa]$ spaces, is proved in Section **??**.

Section **??** discusses angles, including the first variation formula and flat triangles.

1.1 (2+2)-point comparison.

1.1.1. (2+2)-point comparison. *An ordered quadruple of points p^1, p^2, x^1, x^2 in a metric space is said to satisfy* (2+2)-point comparison *(or* (2+2)-point κ-comparison *if a confusion may arise) in the following three cases:*

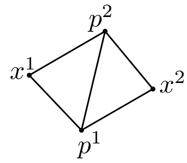
a) $\tilde\measuredangle^\kappa(p^1\,{}^{x^1}_{x^2}) \leqslant \tilde\measuredangle^\kappa(p^1\,{}^{p^2}_{x^1}) + \tilde\measuredangle^\kappa(p^1\,{}^{p^2}_{x^2})$*, or*

b) $\tilde\measuredangle^\kappa(p^2\,{}^{x^1}_{x^2}) \leqslant \tilde\measuredangle^\kappa(p^2\,{}^{p^1}_{x^1}) + \tilde\measuredangle^\kappa(p^2\,{}^{p^1}_{x^2})$*, or*

c) one of the six model angles

$$\tilde\measuredangle^\kappa(p^1\,{}^{x^1}_{x^2}),\ \tilde\measuredangle^\kappa(p^1\,{}^{p^2}_{x^1}),\ \tilde\measuredangle^\kappa(p^1\,{}^{p^2}_{x^2}),\ \tilde\measuredangle^\kappa(p^2\,{}^{x^1}_{x^2}),\ \tilde\measuredangle^\kappa(p^2\,{}^{p^1}_{x^1}),\ \tilde\measuredangle^\kappa(p^2\,{}^{p^1}_{x^2})$$

is undefined.



Spaces in which any two points at distance $< \varpi^\kappa$ are joined by a geodesic and which satisfy (2+2)-point comparisons globally will be called CAT$[\kappa]$ spaces. This terminology was introduced by Gromov; CAT stands for Cartan–Alexandrov–Toponogov, although Toponogov was never doing curvature bounded above. Before Gromov's invention of CAT, these spaces were called $\mathfrak{R}_\kappa$ domains.

1.1.2. Definition. *A ϖ^κ-geodesic space $\mathcal{U}$ is called a* CAT$[\kappa]$ space *(also written $\mathcal{U} \in \mathrm{CAT}[\kappa]$) if every quadruple p^1, p^2, x^1, x^2 satisfies the (2+2)-point κ-comparison (**??**).*

We denote the complete CAT$[\kappa]$ *spaces by* $\overline{\mathrm{CAT}}[\kappa]$.

Theorem **??** gives an alternative definition of $\overline{\mathrm{CAT}}[\kappa]$ spaces.

The condition $\mathcal{U} \in \mathrm{CAT}[\kappa]$ should be thought of as "$\mathcal{U}$ has global curvature $\leqslant \kappa$". In Proposition **??**, we shown that this formulation makes sense; in particular, $\mathrm{CAT}[\kappa] \subset \mathrm{CAT}[K]$ if $\kappa \leqslant K$.

We write $\mathcal{U} \in \mathrm{CAT}$ to mean: $\mathcal{U} \in \mathrm{CAT}[\kappa]$ for some κ.

The following definition gives an infinitesimal notion of upper curvature bound at a point of a space. Initially we shall investigate CAT$[\kappa]$ spaces, turning only in Section **??** to spaces with curvature bounded above.

1.1.3. Definition. *Let $\mathcal{U}$ be a metric space and $p \in \mathcal{U}$. We say that $\mathcal{U}$* has curvature $\leqslant \kappa$ at p *(briefly, $\mathrm{curv}_p \mathcal{U} \leqslant \kappa$) if for any $K > \kappa$ there is $r > 0$ such that $\mathrm{B}[p, r] \in \mathrm{CAT}[K]$.*

We say that a space $\mathcal{U}$ has curvature $\leqslant \kappa$ (briefly, $\mathrm{curv}\,\mathcal{U} \leqslant \kappa$) if $\mathrm{curv}_p \mathcal{U} \leqslant \kappa$ for any $p \in \mathcal{U}$.

If $\mathcal{U} \in \mathrm{CAT}[\kappa]$ then $\mathrm{curv}\,\mathcal{U} \leqslant \kappa$; this fact follows from Proposition **??**.

The converse, namely that an infinitesimal curvature bound implies a global curvature bound, does not hold in general. This failure of the converse for spaces of curvature bounded above is one of the major differences with spaces of curvature bounded below. For example, $\mathbb{S}^1$ is locally isometric to $\mathbb{R}$, and so $\mathrm{curv}\,\mathbb{S}^1 \leqslant 0$, but it is easy to find a quadruple of points in $\mathbb{S}^1$ which violates (2+2)-point comparison.

However, it is true that an infinitesimal bound always implies a local bound; that is, if $\mathrm{curv}\,\mathcal{U} \leqslant \kappa$, then for any p there is $r > 0$ such that $\mathrm{B}[p, r] \in \mathrm{CAT}[\kappa]$ (Corollary **??**).

The next lemma follows directly from Definition **??** and the definitions of ultralimit and ultrapower (see Section **??**).

1.1.4. Proposition. *If $(\mathcal{U}_n, \star_n) \to (\mathcal{U}_\omega, \star_\omega)$ and $\kappa_n \to \kappa_\omega$ as $n \to \omega$, then*

$$\mathcal{U}_n \in \mathrm{CAT}[\kappa_n] \implies \mathcal{U}_\omega \in \overline{\mathrm{CAT}}[\kappa_\omega].$$

Moreover, for a complete metric space $\mathcal{U}$,

$$\mathcal{U} \in \overline{\mathrm{CAT}}[\kappa] \iff \mathcal{U}^\omega \in \overline{\mathrm{CAT}}[\kappa].$$

1.1.5. Unique geodesics. *In a* CAT$[\kappa]$ *space, pairs of points at distance $< \varpi^\kappa$ are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Proof. Let $\mathcal{U} \in \mathrm{CAT}[\kappa]$, $p^1, p^2 \in \mathcal{U}$ and $|p^1 p^2| < \varpi^\kappa$. Suppose $p^1_n \to p^1$, $p^2_n \to p^2$ as $n \to \infty$. Let z_n be the midpoint of a geodesic $[p^1_n p^2_n]$ and z be the midpoint of a geodesic $[p^1 p^2]$.

It suffices to show that

❶ $$|z_n\, z| \to 0 \quad \text{as} \quad n \to \infty.$$

By the triangle inequality, the z_n are approximate midpoints of p^1 and p^2. Apply (2+2)-point comparison (**??**) to the quadruple p^1, p^2, z_n, z. For $p = p^1$ or $p = p^2$, we see that $\tilde\measuredangle^\kappa(p\,{}^{z_n}_{z})$ is arbitrarily small when n is sufficiently large. By the law of cosines, **??** follows. ☐

The following theorem is dual to Plaut's theorem (**??**).

1.1.6. Theorem. *Let $\mathcal{U}$ be a complete ϖ^κ-intrinsic space in which every quadruple p^1, p^2, x^1, x^2 satisfies the (2+2)-point comparison (**??**). Then $\mathcal{U} \in \overline{\mathrm{CAT}}[\kappa]$ (that is, any two points at distance $< \varpi^\kappa$ are joined by a geodesic).*

Proof. Choose a sequence of approximate midpoints z_n for p^1 and p^2; i.e.

❷ $$|p^1\, z_n|, |p^2\, z_n| \to \tfrac{1}{2}\cdot|p^1\, p^2| \text{ as } n \to \infty.$$

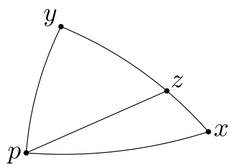
By the law of cosines, $\tilde\measuredangle^\kappa(p^1\,{}^{z_n}_{p^2})$ and $\tilde\measuredangle^\kappa(p^2\,{}^{z_n}_{p^1})$ are arbitrarily small when n is sufficiently large.

Let us apply (2+2)-point comparison (**??**) to the quadruple $p^1, p^2, z_n, z_{\bar{k}}$ with large n and $\bar{k}$. We conclude that $\tilde\measuredangle^\kappa(p\,{}^{z_n}_{z_{\bar{k}}})$ is arbitrarily small when $n, \bar{k}$ are sufficiently large and p is either p^1 or p^2. By **??** and the law of cosines, (z_n) is a Cauchy sequence. Since $\mathcal{U}$ is complete, then (z_n) converges to a midpoint for p^1 and p^2. Thus the existence of a geodesic $[p^1 p^2]$ follows from Lemma **??**. ☐

1.1.7. Corollary. *The completion $\mathcal{U}^c$ of $\mathcal{U} \in \mathrm{CAT}[\kappa]$ satisfies $\mathcal{U}^c \in \overline{\mathrm{CAT}}[\kappa]$.*

Proof. By Theorem **??**, it is only necessary to verify that $\mathcal{U}^c$ is ϖ^κ-intrinsic; this is straightforward. ☐

1.2 More comparisons.



Here we give a few reformulations of Definition **??**.

1.2.1. Theorem. *Let $\mathcal{U}$ be a ϖ^κ-geodesic space. Then*

a) $\mathcal{U} \in \mathrm{CAT}[\kappa]$

if and only if one of the following conditions holds for all $p, x, y \in \mathcal{U}$ of perimeter $< 2\cdot\varpi^\kappa$:

b) (adjacent-angles comparison) for any geodesic $[xy]$ and $z \in]xy[$, we have

$$\tilde\measuredangle^\kappa(z\,{}^{p}_{x}) + \tilde\measuredangle^\kappa(z\,{}^{p}_{y}) \geqslant \pi.$$

c) (point-on-side comparison) for any geodesic $[xy]$ and $z \in]xy[$, we have

$$\tilde\measuredangle^\kappa(x\,{}^{p}_{y}) \geqslant \tilde\measuredangle^\kappa(x\,{}^{p}_{z}),$$

or equivalently,

$$|\tilde p\, \tilde z| \geqslant |p\, z|,$$

where $[\tilde p \tilde x \tilde y] = \tilde\triangle^\kappa(pxy)$, $\tilde z \in]\tilde x \tilde y[$, $|\tilde x\, \tilde z| = |x\, z|$.

d) (angle comparison) for any hinge $[x\,{}^{p}_{y}]$, the angle $\measuredangle[x\,{}^{p}_{y}]$ exists and

$$\measuredangle[x\,{}^{p}_{y}] \leqslant \tilde\measuredangle^\kappa(x\,{}^{p}_{y}),$$

or equivalently,

$$\tilde\Upsilon^\kappa[x\,{}^{p}_{y}] \leqslant |p\,y|.$$

Remark. In the following proof, the part (**??**)⇒(**??**) only requires that the (2+2)-point comparison (**??**) hold for any quadruple, and does not require the existence of geodesics at distance $< \varpi^\kappa$. The same is true of the parts (**??**)⇔(**??**) and (**??**)⇒(**??**). Thus the conditions (**??**), (**??**) and (**??**) are valid for any metric space (not necessarily intrinsic) which satisfies (2+2)-point comparison (**??**). The converse does not hold; for example, all these conditions are vacuously true in a totally disconnected space, while (2+2)-point comparison is not.

*Proof. (**??**) ⇒ (**??**).* Since the perimeter of p, x, y is $< 2\cdot\varpi^\kappa$, so is the perimeter of any subtriple of p, z, x, y by the triangle inequality. By Alexandrov's lemma (**??**),

$$\tilde\measuredangle^\kappa(p\,{}^{z}_{x}) + \tilde\measuredangle^\kappa(p\,{}^{z}_{y}) < \tilde\measuredangle^\kappa(p\,{}^{x}_{y}) \quad\text{or}\quad \tilde\measuredangle^\kappa(z\,{}^{p}_{x}) + \tilde\measuredangle^\kappa(z\,{}^{p}_{y}) = \pi.$$

In the former case, (2+2)-point comparison (**??**) applied to the quadruple p, z, x, y implies

$$\tilde\measuredangle^\kappa(z\,{}^{p}_{x}) + \tilde\measuredangle^\kappa(z\,{}^{p}_{y}) \geqslant \tilde\measuredangle^\kappa(z\,{}^{x}_{y}) = \pi.$$

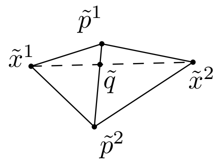
*(**??**) ⇔ (**??**).* Follows directly from Alexandrov's lemma (**??**).

*(**??**) ⇒ (**??**).* By (**??**), for $\bar p \in\,]xp]$ and $\bar y \in\,]xy]$ the function $(|x\bar p|, |x\bar y|) \mapsto$ $\mapsto \tilde\measuredangle^\kappa(x\,{}^{\bar p}_{\bar y})$ is nondecreasing in each argument. In particular, $\measuredangle[x\,{}^{p}_{y}] = \inf \tilde\measuredangle^\kappa(x\,{}^{\bar p}_{\bar y})$. Thus $\measuredangle[x\,{}^{p}_{y}]$ exists and is at most $\tilde\measuredangle^\kappa(x\,{}^{p}_{y})$.

*(**??**) ⇒ (**??**).* By (**??**) and the triangle inequality for angles (**??**),

$$\tilde\measuredangle^\kappa(z\,{}^{p}_{x}) + \tilde\measuredangle^\kappa(z\,{}^{p}_{y}) \geqslant \measuredangle[z\,{}^{p}_{x}] + \measuredangle[z\,{}^{p}_{y}] \geqslant \pi.$$

*(**??**) ⇒ (**??**).* Given a quadruple p^1, p^2, x^1, x^2 whose subtriples have perimeter $< 2\cdot\varpi^\kappa$, we must verify (2+2)-point comparison (**??**). In $\mathbb{M}^2[\kappa]$, construct the model triangles $[\tilde p^1\tilde p^2\tilde x^1] = \tilde\triangle^\kappa(p^1p^2x^1)$ and $[\tilde p^1\tilde p^2\tilde x^2] = \tilde\triangle^\kappa(p^1p^2x^2)$, lying on either side of a common segment $[\tilde p^1\tilde p^2]$. We may suppose



$$\tilde\measuredangle^\kappa(p^1\,{}^{p^2}_{x^1}) + \tilde\measuredangle^\kappa(p^1\,{}^{p^2}_{x^2}) \leqslant \pi \ \text{ and } \ \tilde\measuredangle^\kappa(p^2\,{}^{p^1}_{x^1}) + \tilde\measuredangle^\kappa(p^2\,{}^{p^1}_{x^2}) \leqslant \pi,$$

since otherwise (2+2)-point comparison holds trivially. Then $[\tilde p^1\tilde p^2]$ and $[\tilde x^1\tilde x^2]$ intersect, say at $\tilde q$.

By assumption, there is a geodesic $[p^1p^2]$. Choose $q \in [p^1p^2]$ corresponding to $\tilde q$; that is, $|p^1\,q| = |\tilde p^1\,\tilde q|$. Then

$$|x^1\,x^2| \leqslant |x^1\,q| + |q\,x^2| \leqslant |\tilde x^1\,\tilde q| + |\tilde q\,\tilde x^2| = |\tilde x^1\,\tilde x^2|,$$

where the second inequality follows from (**??**). Therefore by monotonicity of the function $a \mapsto \tilde\measuredangle^\kappa\{a; b, c\}$ (**??**),

$$\tilde\measuredangle^\kappa(p^1\,{}^{x^1}_{x^2}) \leqslant \measuredangle[\tilde p^1\,{}^{\tilde x^1}_{\tilde x^2}] = \tilde\measuredangle^\kappa(p^1\,{}^{p^2}_{x^1}) + \tilde\measuredangle^\kappa(p^1\,{}^{p^2}_{x^2}).$$

□

As a corollary, we display important information from the proof of **??**, namely, monotonicity of the model angle with respect to adjacent sidelengths.

1.2.2. Angle-sidelength monotonicity. *Suppose $\mathcal{U} \in \mathrm{CAT}[\kappa]$, and $p, x, y \in \in \mathcal{U}$ have perimeter $< 2\cdot\varpi^\kappa$. Then for $\bar{y} \in]xy]$, the function*

$$|x\,\bar{y}| \mapsto \tilde{\measuredangle}^\kappa(x\,{}^{p}_{\bar{y}})$$

is nondecreasing.

In particular, if $\bar{p} \in]xp]$ then

a) the function

$$(|x\,\bar{y}|, |x\,\bar{p}|) \mapsto \tilde{\measuredangle}^\kappa(x\,{}^{\bar{p}}_{\bar{y}})$$

is nondecreasing in each argument,

b) the angle $\measuredangle[x\,{}^{p}_{y}]$ exists and

$$\measuredangle[x\,{}^{p}_{y}] = \inf\left\{\, \tilde{\measuredangle}^\kappa(x\,{}^{\bar{p}}_{\bar{y}}) \;\middle|\; \bar{p} \in]xp]\,,\ \bar{y} \in]xy] \,\right\}.$$

1.2.3. Proposition. *If $\kappa < K$, then*

$$\mathrm{CAT}[\kappa] \subset \mathrm{CAT}[K].$$

Moreover

$$\mathrm{CAT}[\kappa] = \bigcap_{K>\kappa} \mathrm{CAT}[K].$$

Proof. The first statement follows from the adjacent-angles comparison (**??**) and the monotonicity of the function $\kappa \mapsto \tilde{\measuredangle}^\kappa(x\,{}^{y}_{z})$ (**??**).

The second statement follows since the function $\kappa \mapsto \tilde{\measuredangle}^\kappa(x\,{}^{y}_{z})$ is continuous. □

1.3 Thin triangles

In this section we introduce the notion of thin triangles, use it to give another definition of CAT spaces, and prove inheritance for thin triangles with respect to decomposition. The inheritance lemma will lead to two fundamental constructions: Alexandrov's patchwork globalization (**??**) and Reshetnyak gluing (**??**).

1.3.1. Definition of κ-thin triangles. *Let $[x^1x^2x^3]$ be a triangle of perimeter $< 2\cdot\varpi^\kappa$ in a metric space. Consider its model triangle $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] = \tilde{\triangle}^\kappa(x^1x^2x^3)$ and the* natural map *$[\tilde{x}^1\tilde{x}^2\tilde{x}^3] \to [x^1x^2x^3]$ that sends a point $\tilde{z} \in [\tilde{x}^i\tilde{x}^j]$ to the corresponding point $z \in [x^ix^j]$ (i.e. such that $|\tilde{x}^i\,\tilde{z}| = |x^i\,z|$ and therefore $|\tilde{x}^j\,\tilde{z}| = |x^j\,z|$).*

We say the triangle $[x^1x^2x^3]$ is κ-thin *if the natural map $[\tilde{x}^1\tilde{x}^2\tilde{x}^3] \to [x^1x^2x^3]$ is short.*

1.3.2. Proposition. *Let $\mathcal{U}$ be a ϖ^κ-geodesic space. Then $\mathcal{U} \in \mathrm{CAT}[\kappa]$ if and only if every triangle of perimeter $< 2\cdot\varpi^\kappa$ in $\mathcal{U}$ is κ-thin.*

Proof. "If" is immediate from point-on-side comparison (**??**). "Only if" follows from Corollary **??**. □

1.3.3. Corollary. *Suppose $\mathcal{U} \in \mathrm{CAT}[\kappa]$. Then any local geodesic in $\mathcal{U}$ of length $< \varpi^\kappa$ is a minimizing geodesic.*

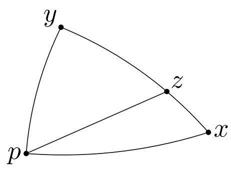
Proof. Suppose $\gamma\colon [0,\ell] \to \mathcal{U}$ is a local geodesic that is not minimizing, with $\ell < \varpi^\kappa$. Choose a to be the maximal value such that γ is minimizing on $[0,a]$. Further choose $b \in (a, \varpi^\kappa - a)$ so that γ is minimizing on $[a,b]$.

Since triangle $[\gamma(0)\gamma(a)\gamma(b)]$ is κ-thin, then

$$|\gamma(a-\varepsilon)\ \gamma(a+\varepsilon)| < 2{\cdot}\varepsilon$$

for all small $\varepsilon > 0$, a contradiction. □

Now let us formulate the main result of this section. The inheritance lemma states that in any metric space, a triangle is κ-thin if it decomposes into κ-thin triangles. In contrast, $\overline{\mathrm{CBB}}[\kappa]$ comparisons are not inherited in this way.



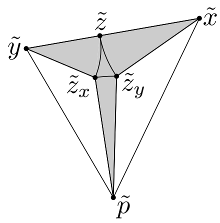
1.3.4. Inheritance lemma for thin triangles. *In a metric space, consider a triangle $[pxy]$ that* decomposes *into two triangles $[pxz]$ and $[pyz]$; that is, $[pxz]$ and $[pyz]$ have common side $[pz]$, and the sides $[xz]$ and $[zy]$ together form the side $[xy]$ of $[pxy]$.*

If triangle $[pxy]$ has perimeter $< 2{\cdot}\varpi^\kappa$ and both triangles $[pxz]$ and $[pyz]$ are κ-thin, then triangle $[pxy]$ is κ-thin.

We shall need the following model-space lemma, which is part of the proof of [**?**, Lemma 2].

1.3.5. Lemma. *Let $[\tilde p\tilde x\tilde y]$ be a triangle in $\mathbb{M}^2[\kappa]$ and $\tilde z \in [\tilde x\tilde y]$. Set $\tilde D = \mathrm{Conv}[\tilde p\tilde x\tilde y]$. Construct points $\dot p, \dot x, \dot z, \dot y \in \mathbb{M}^2[\kappa]$ such that $|\dot p\dot x| = |\tilde p\tilde x|$, $|\dot p\dot y| = |\tilde p\tilde y|$, $|\dot x\dot z| = |\tilde x\tilde z|$, $|\dot y\dot z| = |\tilde y\tilde z|$, $|\dot p\dot z| \leqslant |\tilde p\tilde z|$ and points $\dot x$ and $\dot y$ lie on either side of $[\dot p\dot z]$. Set $\dot D = \mathrm{Conv}[\dot p\dot x\dot z] \cup \mathrm{Conv}[\dot p\dot y\dot z]$.*

Then there is a short map $F\colon \tilde D \to \dot D$ that maps $\tilde p$, $\tilde x$, $\tilde y$ and $\tilde z$ to $\dot p$, $\dot x$, $\dot y$ and $\dot z$ respectively.



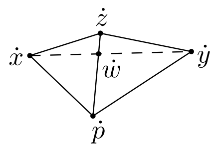
Proof. By Alexandrov's lemma (**??**), there are nonoverlapping triangles $[\tilde p\tilde x\tilde z_y] \overset{iso}{=\!=} [\dot p\dot x\dot z]$ and $[\tilde p\tilde y\tilde z_x] \overset{iso}{=\!=} [\dot p\dot y\dot z]$ inside triangle $[\tilde p\tilde x\tilde y]$.

Connect points in each pair $(\tilde z, \tilde z_x)$, $(\tilde z_x, \tilde z_y)$ and $(\tilde z_y, \tilde z)$ with arcs of circles centered at $\tilde y$, $\tilde p$, and $\tilde x$ respectively. Define F as follows.

- ⋄ Map $\mathrm{Conv}[\tilde p\tilde x\tilde z_y]$ isometrically onto $\mathrm{Conv}[\dot p\dot x\dot y]$; similarly map $\mathrm{Conv}[\tilde p\tilde y\tilde z_x]$ onto $\mathrm{Conv}[\dot p\dot y\dot z]$.
- ⋄ If x is in one of the three circular sectors, say at distance r from center of the circle, let $F(x)$ be the point on the corresponding segment $[pz]$, $[xz]$ or $[yz]$ whose distance from the lefthand endpoint of the segment is r.
- ⋄ Finally, if x lies in the remaining curvilinear triangle $\tilde z\tilde z_x\tilde z_y$, set $F(x) = z$.

By construction, F satisfies the conditions of the lemma. □

*Proof of Inheritance for thin triangles (**??**).* Construct model triangles $[\dot p\dot x\dot z] = \tilde\triangle^\kappa(pxz)$ and $[\dot p\dot y\dot z] = \tilde\triangle^\kappa(pyz)$ so that $\dot x$ and $\dot y$ lie on opposite sides of $[\dot p\dot z]$.



Suppose

$$\tilde\measuredangle^\kappa(z\,{}^p_x) + \tilde\measuredangle^\kappa(z\,{}^p_y) < \pi.$$

Then for some point $\dot w \in [\dot p \dot z]$, we have

$$|\dot x \dot w| + |\dot w \dot y| < |\dot x \dot z| + |\dot z \dot y| = |x y|.$$

Let $w \in [pz]$ correspond to $\dot w$; i.e. $|z w| = |\dot z \dot w|$. Since $[pxz]$ and $[pyz]$ are κ-thin, we have

$$|x w| + |w y| < |x y|,$$

contradicting the triangle inequality.

Thus

$$\tilde\measuredangle^\kappa(z\,{}^p_x) + \tilde\measuredangle^\kappa(z\,{}^p_y) \geqslant \pi.$$

By Alexandrov's lemma (**??**), this is equivalent to

❶ $$\tilde\measuredangle^\kappa(x\,{}^p_z) \leqslant \tilde\measuredangle^\kappa(x\,{}^p_y).$$

Let $[\tilde p \tilde x \tilde y] = \tilde\triangle^\kappa(pxy)$ and $\tilde z \in [\tilde x \tilde y]$ correspond to z; i.e. $|x z| = |\tilde x \tilde z|$. Inequality **??** is equivalent to $|p z| \leqslant |\tilde p \tilde z|$. Hence Lemma **??** applies. Therefore there is a short map F that sends $[\tilde p \tilde x \tilde y]$ to $\dot D = \mathrm{Conv}[\dot p \dot x \dot z] \cup \mathrm{Conv}[\dot p \dot y \dot z]$ in such a way that $\tilde p \mapsto \dot p$, $\tilde x \mapsto \dot x$, $\tilde z \mapsto \dot z$ and $\tilde y \mapsto \dot y$.

By assumption, the natural maps $[\dot p \dot x \dot z] \to [pxz]$ and $[\dot p \dot y \dot z] \to [pyz]$ are short. By composition, the natural map from $[\tilde p \tilde x \tilde y]$ to $[pyz]$ is short, as claimed. □

1.4 Function comparison and development

In this section, we give analytic and geometric ways of viewing the point-on-side comparison (**??**) as a convexity condition.

First we obtain a corresponding differential inequality for the distance function in $\mathcal U$. In particular, a geodesic space $\mathcal U$ is in $\mathrm{CAT}[0]$ if and only if for any $p \in \mathcal U$, the function $|p*|^2 : \mathcal U \to \mathbb R$ is 2-convex; see Section **??** for definition.

1.4.1. Theorem. *Suppose $\mathcal U$ is a ϖ^κ-geodesic space. Then the following are equivalent:*

a) $\mathcal U \in \mathrm{CAT}[\kappa]$,

b) for any $p \in \mathcal U$, the function $f = \mathrm{md}^\kappa \circ |p*|$ satisfies $f'' + \kappa \cdot f \geqslant 1$ in $\mathrm B(p, \varpi^\kappa)$.

Proof. For a geodesic $[xy]$ in $\mathrm B(p, \varpi^\kappa)$, consider the model triangle $[\tilde p \tilde x \tilde y] = \tilde\triangle^\kappa(pxy)$, and set

$$\tilde r(t) = |\tilde p\, \mathrm{geod}_{[\tilde x \tilde y]}(t)|, \qquad r(t) = |p\, \mathrm{geod}_{[xy]}(t)|.$$

Set $\tilde f = \mathrm{md}^\kappa \circ \tilde r$ and $f = \mathrm{md}^\kappa \circ r$. By Property **??**, we have $\tilde f'' = 1 - \kappa \cdot \tilde f$. Clearly $\tilde f(t)$ and $f(t)$ agree at $t = 0$ and $t = |xy|$. The point-on-side comparison (**??**) is the condition $r(t) \leqslant \tilde r(t)$ for all $t \in [0, |xy|]$. Since md^κ is increasing on $[0, \varpi^\kappa)$, $r \leqslant \tilde r$ and $f \leqslant \tilde f$ are equivalent. Thus the claim follows by Jensen's inequality (**??**). □

1.4.2. Corollary. *Suppose $\mathcal U \in \mathrm{CAT}[\kappa]$. Then:*

a) Any open ball of radius $R \leqslant \varpi^\kappa/2$ in $\mathcal{U}$ is convex.
b) Any closed ball of radius $R < \varpi^\kappa/2$ in $\mathcal{U}$ is convex.
c) If $\mathcal{U} \in \overline{\mathrm{CAT}}[\kappa]$, any closed ball of radius $R = \varpi^\kappa/2$ in $\mathcal{U}$ is weakly convex.

Proof. Suppose $p \in \mathcal{U}$, $R \leqslant \varpi^\kappa/2$ and $x, y \in \mathrm{B}(p, R)$. By the triangle inequality, $|xy| < \varpi^\kappa$ and so any geodesic $[xy]$ lies in $\mathrm{B}(p, \varpi^\kappa)$. The function comparison (**??**) gives $[xy] \subset \mathrm{B}(p, R)$. Hence (**??**).

The part (**??**) follows since for any $R < \varpi^\kappa/2$,

$$\overline{\mathrm{B}}[p, R] = \bigcap_{R < \bar{R} \leqslant \varpi^\kappa/2} \mathrm{B}(p, \bar{R}).$$

Finally, since $\mathcal{U}$ is a locally intrinsic space, $\overline{\mathrm{B}}[p, \varpi^\kappa/2]$ is the Hausdorff limit of $\overline{\mathrm{B}}[p, R]$ for $R \to \varpi^\kappa/2-$. Hence (**??**) follows from **??**. □

Geometrically, the development construction (**??**) translates distance comparison into a local convexity statement for subsets of $\mathbb{M}^2[\kappa]$. Recall that a curve in $\mathbb{M}^2[\kappa]$ is *(locally) concave with respect to p* if (locally) its supergraph with respect to p is a convex subset of $\mathbb{M}^2[\kappa]$ (**??**).

1.4.3. Development criterion. *For a ϖ^κ-geodesic space $\mathcal{U}$, the following statements hold:*
a) For any $p \in \mathcal{U}$ and any geodesic $\gamma\colon [0, T] \to \mathrm{B}(p, \varpi^\kappa)$, suppose the κ-development $\tilde\gamma$ in $\mathbb{M}^2[\kappa]$ of γ with respect to p is locally concave. Then $\mathcal{U} \in \mathrm{CAT}[\kappa]$.
b) If $\mathcal{U} \in \mathrm{CAT}[\kappa]$, then for any $p \in \mathcal{U}$ and any geodesic $\gamma : [0, T] \to \to \mathcal{U}$ for which the triangle $[p\gamma(0)\gamma(T)]$ has perimeter $< 2{\cdot}\varpi^\kappa$, the κ-development $\tilde\gamma$ in $\mathbb{M}^2[\kappa]$ of γ with respect to p is concave.

*Proof. (**??**).* Set $\gamma = \mathrm{geod}_{[xy]}$ and $T = |xy|$. Let $\tilde\gamma\colon [0, T] \to \mathbb{M}^2[\kappa]$ be the concave κ-development based at $\tilde p$ of γ with respect to p. Let us show that the function

$$❶ \qquad t \mapsto \tilde\measuredangle^\kappa(x\,{}^{p}_{\gamma(t)})$$

is nondecreasing.

For a partition $0 = t^0 < t^1 < \ldots < t^n = T$, set

$$\tilde y^i = \tilde\gamma(t^i) \ \text{ and } \ \tau^i = |\tilde y^0\,\tilde y^1| + |\tilde y^1\,\tilde y^2| + \ldots + |\tilde y^{i-1}\,\tilde y^i|.$$

Since $\tilde\gamma$ is locally concave, for a sufficiently fine partition the broken geodesic $\tilde y^0\tilde y^1\ldots\tilde y^n$ is locally convex with respect to $\tilde p$. Alexandrov's lemma (**??**), applied inductively to pairs of triangles $\tilde\triangle^\kappa\{\tau^{i-1}, |p\tilde y^0|, |p\tilde y^{i-1}|\}$ and $\tilde\triangle^\kappa\{|\tilde y^{i-1}\,\tilde y^i|, |p\tilde y^{i-1}|, |p\tilde y^i|\}$, shows that the sequence $\tilde\measuredangle^\kappa\{|p\tilde y^i|; |p\tilde y^0|, \tau^i\}$ is nondecreasing.

Taking finer partitions and passing to the limit,

$$\max_i\{|\tau^i - t^i|\} \to 0,$$

we get **??** and the point-on-side comparison (**??**) follows.

(??). Consider a partition $0 = t^0 < t^1 < \ldots < t^n = T$, and set $x^i = \gamma(t^i)$. Construct a chain of model triangles $[\tilde p\tilde x^{i-1}\tilde x^i] = \tilde\triangle^\kappa(px^{i-1}x^i)$ with the direction of $[\tilde p\tilde x^i]$ turning counterclockwise as i grows. By angle comparison (??),

❷ $$\measuredangle[\tilde x^i\,{}^{\tilde x^{i-1}}_{\tilde p}] + \measuredangle[\tilde x^i\,{}^{\tilde x^{i+1}}_{\tilde p}] \geqslant \pi.$$

Since γ is a geodesic,

❸ $$\operatorname{length}\gamma = \sum_{i=1}^n |x^{i-1}\,x^i| \leqslant |px^0| + |px^n|.$$

By repeated application of Alexandrov's Lemma (??), and inequality **??**,

$$\sum_{i=1}^n \measuredangle[\tilde p\,{}^{\tilde x^{i-1}}_{\tilde x^i}] \leqslant \tilde\measuredangle^\kappa(p\,{}^{x^0}_{x^n}) \leqslant \pi.$$

Then by **??**, the broken geodesic $\tilde p\tilde x^0\tilde x^1\ldots\tilde x^n$ is concave with respect to $\tilde p$.

By Lemma **??** applied to the polygons $[\tilde x^0\tilde x^1\ldots\tilde x^n]$, the broken geodesics $\tilde x^0\tilde x^1\ldots\tilde x^n$ approach the development of γ with respect to p. Hence the result. □

1.5 Alexandrov's patchwork globalization

The following theorem, which essentially is [?, Satz 9], gives a global condition on geodesics that is necessary and sufficient for a space $\mathcal U$ with $\operatorname{curv}\mathcal U \leqslant \kappa$ to satisfy $\mathcal U \in \mathrm{CAT}[\kappa]$. The proof use a thin-triangle decompositions, and the inheritance lemma (??).

1.5.1. Patchwork globalization theorem. *For a metric space $\mathcal U$, the following two statements are equivalent:*

a) $\mathcal U \in \mathrm{CAT}[\kappa]$.

b) $\operatorname{curv}\mathcal U \leqslant \kappa$*; moreover, pairs of points in $\mathcal U$ at distance $< \varpi^\kappa$ are joined by unique geodesics, and these geodesics depend continuously on their endpoint pairs.*

Note that the implication (??)⇒(??) is immediate, being the statement of Lemma **??**.

The following corollary is immediate. (Recall that convex and weakly convex sets are defined in Section **??**.)

1.5.2. Corollary. *In a* $\mathrm{CAT}[\kappa]$ *space, any weakly ϖ^κ-convex set is ϖ^κ-convex.*

The next corollary states that an infinitesimal curvature bound in a neighborhood of a point implies a local curvature bound.

1.5.3. Corollary. *Let $\mathcal U$ be a metric space and $\Omega \subset \mathcal U$ be an open subset. Assume* $\operatorname{curv}_p\mathcal U \leqslant \kappa$ *for any $p \in \Omega$. Then for any point $p \in \Omega$ there is $R > 0$ such that* $\mathrm B(p,R)$ *is a convex subset of $\mathcal U$ and* $\mathrm B(p,R) \in \mathrm{CAT}[\kappa]$.

Proof. Fix $K > \kappa$. According to Definition **??**, there is $R > 0$ such that $\mathrm B(p,R) \in \mathrm{CAT}[K]$. We may assume that $\mathrm B(p,R) \subset \Omega$ and $R < \varpi^K/2$. According to Corollary **??**, we may assume that

◇ $R < \varpi^K/2$,
◇ $\mathrm{B}(p,R)$ is convex,
◇ $\mathrm{B}(p,R) \subset \Omega$.

In particular $\operatorname{curv} \mathrm{B}(p,R) \leqslant \kappa$.

According to (**??**)⇒(**??**) of Patchwork globalization (**??**), any two points in $\mathrm{B}(p,R)$ can be joined by a unique geodesic, which depends continuously on its endpoints.

Then(**??**)⇒(**??**) of Patchwork globalization implies $\mathrm{B}(p,R) \in \mathrm{CAT}\lceil\kappa\rceil$. □

The proof of Patchwork globalization uses the following construction:

1.5.4. Line-of-sight map. *Let $\mathcal{U}$ be a metric space in which pairs of points at distance $< \varpi^\kappa$ are joined by unique geodesics and these geodesics depend continuously on their endpoint pairs.*

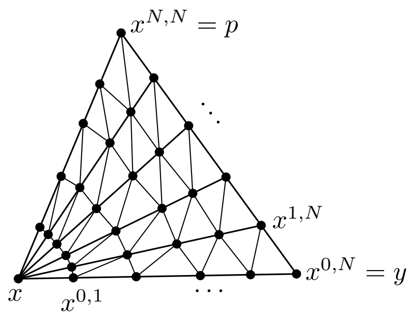
Consider a point p and a curve $\bar\gamma$ of finite length in $\mathrm{B}(p,\varpi^\kappa)$, and let $\gamma\colon [0,1] \to \mathcal{U}$ and $\gamma_t\colon [0,1] \to \mathcal{U}$ be the constant-speed parameterizations of $\bar\gamma$ and the geodesics $[p\gamma(t)]$ respectively. Then the map

$$[0,1]\times[0,1] \to \mathcal{U}\colon (t,s) \mapsto \gamma_t(s)$$

is continuous; it will be called the line-of-sight map *for $\bar\gamma$ from p.*

*Proof of Patchwork globalization theorem (**??**).* As was already noted, it only remains to prove (**??**)⇒(**??**).

Fix arbitrary $K > \kappa$. Let $[pxy]$ be a triangle of perimeter $< 2\cdot\varpi^\kappa$ in $\mathcal{U}$. According to propositions **??** and **??**, it is sufficient to show the triangle $[pxy]$ is K-thin.



Consider the line-of-sight map for $[xy]$ from p. For a partition

$$0 = t^0 < t^1 < \ldots < t^N = 1,$$

set $x^{i,j} = \gamma_{t^i}(t^j)$. Since the line-of-sight map is continuous, by definition **??** we may assume each triangle $[x^{i,j}\, x^{i,j+1}\, x^{i+1,j+1}]$ and $[x^{i,j}\, x^{i+1,j}\, x^{i+1,j+1}]$ is K-thin (see Proposition **??**).

Now we show that the K-thin property propagates to $[pxy]$ by repeated application of the inheritance lemma (**??**):

◇ First, for fixed i, sequentially applying the lemma shows that the triangles $[x\, x^{i,1}\, x^{i+1,2}]$, $[x\, x^{i,2}\, x^{i+1,2}]$, $[x\, x^{i,2}\, x^{i+1,3}]$, and so on are K-thin.

In particular, for each i, the long triangle $[x\, x^{i,N}\, x^{i+1,N}]$ is K-thin.

◇ Applying the lemma again shows that the triangles $[x\, x^{0,N}\, x^{2,N}]$, $[x\, x^{0,N}\, x^{3,N}]$, and so on are K-thin.

In particular, $[pxy] = [p\, x^{0,N}\, x^{N,N}]$ is K-thin. □

1.6 Space of geodesics

In this section we prove a "no-conjugate-point" theorem for spaces with upper curvature bounds, and derive from it a number of statements about local geodesics. These statements will be used in the proof of the Hadamard–Cartan theorem (**??**) and Lifting globalization theorem (**??**), in much the same way as the exponential map is used in Riemannian geometry.

1.6.1. Proposition. *Let $\mathcal{U}$ be a metric space with* $\operatorname{curv}\mathcal{U}\leqslant\kappa$. *Let* $\gamma_n\colon[0,1]\to\to\mathcal{U}$ *be a sequence of local geodesic paths converging to a path* $\gamma_\infty\colon[0,1]\to\mathcal{U}$. *Then* γ_∞ *is a local geodesic path. Moreover*

$$\operatorname{length}\gamma_n\to\operatorname{length}\gamma_\infty.$$

Proof. Fix $t\in[0,1]$. By Corollary **??**, we may choose R satisfying $0<R<\varpi^K$, and such that the ball $\mathcal{B}=\mathrm{B}(\gamma_\infty(t),R)$ is a convex subset of $\mathcal{U}$ and forms a $\mathrm{CAT}[\kappa]$ space.

A local geodesic segment with length $<R/2$, and intersecting $\mathrm{B}(\gamma_\infty(t),R/2)$, cannot leave $\mathcal{B}$ and hence is minimizing by Corollary **??**. In particular, for all sufficiently large n, any subsegment of γ_n through $\gamma_n(t)$ with length $<R/2$ is a geodesic.

Since $\mathcal{B}$ is a $\mathrm{CAT}[\kappa]$ space, geodesic segments in $\mathcal{B}$ depend uniquely and continuously on their endpoint pairs by Lemma **??**. Thus there is a subinterval $\mathbb{I}$ of $[0,1]$, which contains a neighborhood of t in $[0,1]$ and such that $\gamma_n|\mathbb{I}$ is minimizing for all large n. It follows that $\gamma_\infty|\mathbb{I}$ is a geodesic, and therefore γ_∞ is a local geodesic. □

By analogy with Riemannian geometry, the main statement of the following theorem could be restated as: *In a space of curvature $\leqslant\kappa$, two points cannot be conjugate along a local geodesic of length $<\varpi^\kappa$.*

The following theorem appears in [**?**].

1.6.2. No-conjugate-point theorem. *Suppose $\mathcal{U}$ is a locally complete metric space with* $\operatorname{curv}\mathcal{U}\leqslant\kappa$. *Let* $\gamma\colon[0,1]\to\mathcal{U}$ *be a local geodesic path with length* $<\varpi^\kappa$. *Then for some neighborhoods $\Omega^0\ni\gamma(0)$ and $\Omega^1\ni\gamma(1)$ there is a unique continuous map*

$$(x,y,t)\mapsto\gamma_{xy}(t)\colon\Omega^0\times\Omega^1\times[0,1]\to\mathcal{U}$$

such that $\gamma_{xy}\colon[0,1]\to\mathcal{U}$ is a local geodesic path with $\gamma_{xy}(0)=x$ and $\gamma_{xy}(1)=y$ for each $(x,y)\in\Omega^0\times\Omega^1$, and the family γ_{xy} contains γ. Moreover:

a) The map

$$(x,y,t)\mapsto\gamma_{xy}(t)\colon\Omega^0\times\Omega^1\times[0,1]\to\mathcal{U}$$

is $\pounds$-Lipschitz for $\pounds=\max\left\{\frac{\operatorname{sn}^\kappa r}{\operatorname{sn}^\kappa\ell}\mid 0\leqslant r\leqslant\ell\right\}$.

b) There is a space $\mathcal{N}\in\overline{\mathrm{CAT}}[\kappa]$, an open set $\hat\Omega\subset\mathcal{N}$, and a locally isometric immersion

$$\mathfrak{m}\colon\hat\Omega\looparrowright\mathcal{U},$$

such that Ω^0 and Ω^1 are isometric images under $\mathfrak{m}$ of open sets $\hat\Omega^0$ and $\hat\Omega^1$ in $\hat\Omega$, and for $\hat x\in\hat\Omega^0$, $\hat y\in\hat\Omega^1$, the geodesic $\hat\gamma_{\hat x\hat y}$ in $\mathcal{N}$ satisfies $\mathfrak{m}\circ\hat\gamma_{\hat x\hat y}=\gamma_{xy}$.

The No-conjugate-point theorem follows from the next lemma, which was suggested to us by Alexander Lytchak. The proof proceeds by piecing together $\mathrm{CAT}[\kappa]$ neighborhoods of points on a curve to construct a new $\overline{\mathrm{CAT}}[\kappa]$ space. For the original proof of Theorem **??**, see Exercise **??**.

1.6.3. Patchwork along a curve. *Let $\mathcal{U}$ be a locally complete metric space with* $\operatorname{curv}\mathcal{U} \leqslant \kappa$ *and* $\alpha\colon [a,b] \to \mathcal{U}$ *be a curve.*

Then there is a space $\mathcal{N} \in \mathrm{CAT}[\kappa]$ *with an open set* $\hat{\Omega} \subset \mathcal{N}$*, a curve* $\hat{\alpha}\colon [0,1] \to \hat{\Omega}$*, and an open locally isometric immersion* $\mathfrak{IM}\colon \hat{\Omega} \looparrowright \mathcal{U}$ *such that* $\mathfrak{IM} \circ \hat{\alpha} = \alpha$.

Moreover if α *is simple, then one can assume in addition that* $\mathfrak{IM}$ *is an open embedding; thus* $\hat{\Omega}$ *is locally isometric to a neighborhood of* $\Omega = \mathfrak{IM}(\hat{\Omega})$ *of* α.

Proof. According to Corollary **??**, for any $t \in [a,b]$ there is $r(t) > 0$ such that the closed ball $\overline{\mathrm{B}}[\alpha(t), r(t)]$ is a convex set that forms a $\overline{\mathrm{CAT}}[\kappa]$ space.

Choose balls $\mathcal{B}^i = \overline{\mathrm{B}}[\alpha(t^i), r(t^i)]$ for some partition $a = t^0 < t^1 < \ldots < t^n = b$ in such a way that $\operatorname{Int}\mathcal{B}^i \supset \alpha([t^{i-1}, t^i])$ for all $i > 0$.

Consider the disjoint union $\bigsqcup_i \mathcal{B}^i = \{\, (i,x) \mid x \in \mathcal{B}^i \,\}$ with the minimal equivalence relation $\sim$ such that $(i,x) \sim (i-1,x)$ for all $i > 0$. Let $\mathcal{N}$ be the space obtained by gluing the $\mathcal{B}^i$ along $\sim$. Note that $A^i = \mathcal{B}^i \cap \mathcal{B}^{i-1}$ is convex in $\mathcal{B}^i$ and in $\mathcal{B}^{i-1}$. Applying the Reshetnyak gluing theorem (**??**) n times, we conclude that $\mathcal{N} \in \overline{\mathrm{CAT}}[\kappa]$.

For $t \in [t^{i-1}, t^i]$, let $\hat{\alpha}(t)$ be the equivalence class of $(i, \alpha(t))$ in $\mathcal{N}$. Let $\hat{\Omega}$ be the ε-neighborhood of $\hat{\alpha}$ in $\mathcal{N}$, where $\varepsilon > 0$ is chosen so that $\mathrm{B}(\alpha(t), \varepsilon) \subset \mathcal{B}^i$ for all $t \in [t^{i-1}, t^i]$.

Define $\mathfrak{IM}\colon \hat{\Omega} \to \mathcal{U}$ by sending the equivalence class of (i,x) to x. It is straightforward to check that $\mathfrak{IM}\colon \mathcal{N} \to \mathcal{U}$, $\hat{\alpha}\colon [0,1] \to \mathcal{N}$ and $\hat{\Omega} \subset \mathcal{N}$ satisfy the conclusion of the main part of the lemma.

To prove the final statement in the lemma, we only have to choose $\varepsilon > 0$ so that in addition $|\alpha(\tau)\ \alpha(\tau')| > 2{\cdot}\varepsilon$ if $\tau \leqslant t^{i-1}$ and $t^i \leqslant \tau'$ for some i. ☐

Proof of No-conjugate-point theorem **??**. Apply patchwork along γ (**??**). ☐

The No-conjugate-point theorem allows us to move a local geodesic path so that its endpoints follow given trajectories. This process might terminate in two cases: when the length of the local geodesic path approaches ϖ^κ, or when the homotopy leaves a complete subset of the space. The following corollary formulates these cases more precisely.

1.6.4. Corollary. *Let $\mathcal{U}$ be a locally complete metric space with* $\operatorname{curv}\mathcal{U} \leqslant \kappa$*. Suppose* $\gamma\colon [0,1] \to \mathcal{U}$ *is a local geodesic path with length* $< \varpi^\kappa$*. Let* $\alpha^i\colon [0,1] \to \mathcal{U}$*, for* $i = 0,1$*, be paths starting at* $\gamma(0)$ *and* $\gamma(1)$ *respectively.*

Then there is a uniquely determined pair consisting of an interval $\mathbb{I}$ *satisfying* $0 \in \mathbb{I} \subset [0,1]$*, and a continuous family of local geodesic paths* $\gamma_t\colon [0,1] \to \mathcal{U}$ *for* $t \in \mathbb{I}$*, such that*

a) $\gamma_0 = \gamma$, $\gamma_t(0) = \alpha_0(t)$, $\gamma_t(1) = \alpha^1(t)$*, and* γ_t *has length* $< \varpi^\kappa$*,*

b) if $\mathbb{I} \neq [0,1]$*, then* $\mathbb{I} = [0,a)$*, where either* γ_t *converges uniformly to a local geodesic* γ_a *of length* ϖ^κ*, or for some fixed* $s \in [0,1]$ *the curve* $\gamma_t(s) : [0,a) \to \mathcal{U}$ *is a Lipschitz curve with no limit as* $t \to a-$.

Proof. Uniqueness follows from Theorem **??**.

Let $\mathbb{I}$ be the maximal interval for which there is a family γ_t satisfying condition (**??**). By Theorem **??**, such an interval exists and is open in $[0,1]$. Suppose $\mathbb{I} \neq [0,1]$. Then $\mathbb{I} = [0,a)$ for some $0 < a \leqslant 1$. It suffices to show that $\mathbb{I}$ satisfies condition (**??**).

For each fixed $s \in [0,1]$, define the curve $\alpha_s : [0,a) \to \mathcal{U}$ by $\alpha_s(t) = \gamma_t(s)$. By Theorem **??**, α_s is $\pounds$-Lipschitz for some $\pounds$.

If α_s for some value of s does not converge as $t \to a-$, then condiion (??) holds. If each α_s converges as $t \to a-$, then γ_t converges as $t \to a-$, say to γ_a. By Proposition ??, γ_a is a local geodesic path and

$$\operatorname{length}\gamma_t \to \operatorname{length}\gamma_a \leqslant \varpi^\kappa.$$

By maximality of $\mathbb{I}$, $\operatorname{length}\gamma_a = \varpi^\kappa$ and so condition (??) again holds. □

1.6.5. Corollary. *Suppose $\mathcal{U}$ is a complete space with $\operatorname{curv}\mathcal{U} \leqslant \kappa$, and $\alpha\colon [0,1] \to \to \mathcal{U}$ is a path of length $\ell < \varpi^\kappa$. Then for all $t \in [0,1]$,*

a) there is unique homotopy of local geodesic paths $\gamma_t\colon [0,1] \to \mathcal{U}$ such that $\gamma_0(t) = \gamma_t(0) = \alpha(0)$ and $\gamma_t(1) = \alpha(t)$,

b)
$$\operatorname{length}\gamma_t \leqslant \operatorname{length}(\alpha|[0,t]),$$
and equality holds for given t if and only if the restriction $\alpha|[0,t]$ is a reparametrization of γ_t.

Proof. Apply Corollary ??, setting $\alpha^0(t) = \alpha(0)$ for all $t \in [0,1]$, and $\alpha^1 = \alpha$. Since $\mathcal{U}$ is complete, there is an interval $\mathbb{I}$ such that statement (??) holds for all $t \in \mathbb{I}$, and either $\mathbb{I} = [0,1]$ or $\mathbb{I} = [0,a)$ where γ_t converges uniformly to a local geodesic γ_a of length ϖ^κ.

Theorem ?? implies that the values of t for which condition (??) holds form an open subset of $\mathbb{I}$ containing 0; clearly this subset is also closed in $\mathbb{I}$. Therefore (??) holds on all of $\mathbb{I}$.

Sinse $\ell < \varpi^\kappa$, $\mathbb{I} = [0,1]$ and the result follows. □

1.7 Lifting globalization

By the Hadamard–Cartan theorem (see ?? below), the universal metric cover of a complete space with curvature $\leqslant 0$ is a $\operatorname{CAT}[0]$ space. The Lifting globalization theorem gives an appropriate generalization of the above statement to arbitrary curvature bounds.

1.7.1. Lifting globalization theorem. *Suppose $\mathcal{U}$ is a complete space with $\operatorname{curv}\mathcal{U} \leqslant \kappa$ and $p \in \mathcal{U}$. Then there is a space $\mathcal{B} \in \operatorname{CAT}[\kappa]$, where $\mathcal{B} = \operatorname{B}[\hat{p}, \varpi^\kappa/2]$ for some $\hat{p} \in \mathcal{B}$, and a locally isometric map $\mathfrak{m}\colon \mathcal{B} \to \mathcal{U}$ with $\mathfrak{m}(\hat{p}) = p$ and the following lifting property: for any path $\alpha\colon [0,1] \to \mathcal{U}$ with $\alpha(0) = p$ and $\operatorname{length}\alpha \leqslant \varpi^\kappa/2$, there is a unique path $\hat{\alpha}\colon [0,1] \to \mathcal{B}$ such that $\hat{\alpha}(0) = \hat{p}$ and $\mathfrak{m} \circ \hat{\alpha} \equiv \alpha$.*

The Hadamard–Cartan theorem for intrinsic spaces $\mathcal{U}$ satisfying $\operatorname{curv}\mathcal{U} \leqslant 0$ comes as a corollary.

The Hadamard–Cartan theorem was first stated for geodesic spaces by Gromov [?, p.119]. A detailed proof of Gromov's statement when $\mathcal{U}$ is proper was given in [?]. The statement in the non-proper case was proved in [?]; this proof applies more generally, to *convex spaces* (see Exercise ??). It was pointed out by Bruce Kleiner (see [?, ?]) and independently by Martin Bridson and André Haefliger [?] that the No-conjugate-point theorem (??) allows curve-shortening as in Corollary ??, and hence the Hadamard–Cartan theorem applies to intrinsic as well as geodesic spaces.

1.7.2. Hadamard–Cartan theorem. *Let $\mathcal{U}$ be a complete, simply connected intrinsic space satisfying* $\operatorname{curv}\mathcal{U} \leqslant 0$. *Then* $\mathcal{U} \in \overline{\mathrm{CAT}}[0]$.

Proof. Since $\varpi^\kappa = \infty$, the map $\mathfrak{m}\colon \mathcal{B} \to \mathcal{U}$ in Theorem (**??**) is a metric covering. Since $\mathcal{U}$ is simply connected, $\mathfrak{m}\colon \mathcal{B} \to \mathcal{U}$ is an isometry. □

The proof of the lifting globalization theorem relies heavily on the properties of the space of local geodesic paths discussed in Section **??**. The following lemma from [**?**] is a key step in the proof.

1.7.3. Radial lemma. *Let $\mathcal{U}$ be an intrinsic space with* $\operatorname{curv}\mathcal{U} \leqslant \kappa$, *and suppose $p \in \mathcal{U}$, $R \leqslant \varpi^\kappa/2$. Assume the ball $\overline{\mathrm{B}}[p,\bar R]$ is complete for all $\bar R < R$, and there is a unique geodesic path,* $\mathrm{path}_{[px]}$, *for any $x \in \mathrm{B}(p,R)$, where* $\mathrm{path}_{[px]}$ *depends continuously on x. Then* $\mathrm{B}(p,R) \in \mathrm{CAT}[\kappa]$.

Proof. Without loss of generality, we may assume $\mathcal{U} = \mathrm{B}(p,\varpi^\kappa)$.

Set $f = \mathrm{md}^\kappa \circ |p{*}|$. Let us show that

❶ $$f'' + \kappa{\cdot}f \geqslant 1.$$

Fix $z \in \mathcal{U}$, and apply Theorem **??** for the unique geodesic path γ from p to z. We use the notations Ω^i, γ_{xy}, $\mathcal{N}$, $\hat x$, $\hat y$, as in that theorem. In particular, $z \in \Omega^1$.

By assumption, $\gamma_{py} = \mathrm{path}_{[py]}$ for any $y \in \Omega^1$. Consequently, $f(y) = \mathrm{md}^\kappa|\hat p\hat y|_{\mathcal{N}}$. From the function comparison (**??**) applied in $\mathcal{N}$, we have $f'' + \kappa{\cdot}f \geqslant 1$ in Ω^1.

Fix $\bar R < R$, and consider the complete closed ball $\overline{\mathrm{B}}[p,\bar R] \subset \mathcal{U}$.

❷ $\overline{\mathrm{B}}[p,\bar R]$ *forms a convex set in $\mathcal{U}$.*

The proof of this claim takes most of the rest of proof of the theorem.

Choose arbitrary $x, z \in \overline{\mathrm{B}}[p,\bar R]$. First note that **??** implies:

❸ *If $\gamma\colon [0,1] \to \mathcal{U}$ is a local geodesic path from x to z and* $\operatorname{length}\gamma < \varpi^\kappa$, *then* $\operatorname{length}\gamma < 2{\cdot}\bar R$ *and γ lies completely in $\overline{\mathrm{B}}[p,\bar R]$.*

Note that $|x\,z| < \varpi^\kappa$. Thus, to prove Claim **??**, it is sufficient to show that there is a geodesic path of $\mathcal{U}$ from x to z. By Corollary **??**,

❹ *Given a path $\alpha\colon [0,1] \to \mathcal{U}$ from x to z with* $\operatorname{length}\alpha < \varpi^\kappa$, *there is a local geodesic path γ from x to z such that*

$$\operatorname{length}\gamma \leqslant \operatorname{length}\alpha.$$

Further, let us prove the following.

❺ *There is unique local geodesic path γ_{xz} in $\overline{\mathrm{B}}[p,\bar R]$ from x to z.*

Denote by Δ_{xz} the set of all local geodesic paths of $\overline{\mathrm{B}}[p,\bar R]$ from x to z. By Corollary **??**, there is a bijection $\Delta_{xz} \to \Delta_{pp}$. According to **??**, Δ_{pp} contains only the constant path. Claim **??** follows.

Note that claims **??**, **??** and **??** imply that γ_{xz} is minimizing; hence Claim **??**.

Further, Claim **??** and the No-conjugate-point theorem (**??**) together imply that the map $(x,z) \mapsto \gamma_{xz}$ is continuous.

Therefore by the Patchwork globalization theorem (**??**), $\overline{\mathrm{B}}[p,\bar R] \in \mathrm{CAT}[\kappa]$.

Since

$$\mathrm{B}(p,R) = \bigcup_{\bar R<R} \overline{\mathrm{B}}[p,\bar R],$$

then $\mathrm{B}(p,R)$ is convex in $\mathcal{U}$ and (2+2)-point comparison holds for any quadruple in $\mathrm{B}(p,R)$. Therefore $\mathrm{B}(p,\varpi^\kappa/2)\in \mathrm{CAT}[\kappa]$. □

In the following proof, we first construct a space Γ out of geodesics that start at p. The space Γ comes with a marked point $\hat p$ and the right-hand-endpoint map $\mathfrak{m}\colon\Gamma\to\mathcal{U}$. In the Riemannian case, one can think of the map $\mathfrak{m}$ as $\exp_p$, and the space Γ as the ball of radius ϖ^κ in the tangent space at p, equipped with the metric pulled back by $\exp_p$.

We are going to set $\mathcal{B}=\mathrm{B}[\hat p,\varpi^\kappa/2]\subset\Gamma$, and use the Radial lemma (**??**) to prove $\mathcal{B}\in\mathrm{CAT}[\kappa]$.

Proof of Lifting globalization theorem **??**. Suppose $\hat\gamma$ is a homotopy of local geodesic paths that start at p. Thus the map

$$\hat\gamma\colon (t,\tau)\mapsto\hat\gamma_t(\tau)\colon[0,1]\times[0,1]\to\mathcal{U}$$

is continuous and for each t,

- $\hat\gamma_t(0)=p$,
- $\hat\gamma_t\colon[0,1]\to\mathcal{U}$ forms a local geodesic path in $\mathcal{U}$.

Let $\vartheta(\hat\gamma)$ denote the length of the path $t\mapsto\hat\gamma_t(1)$.

Let Γ be the set of all geodesic paths with length $<\varpi^\kappa$ in $\mathcal{U}$ that start at p. Let $\hat p\in\Gamma$ be the constant path $\hat p(t)\equiv p$. Given $\alpha,\beta\in\Gamma$ define

$$|\alpha\,\beta|_\Gamma=\inf\{\vartheta(\hat\gamma)\},$$

with infimum taken along all homotopies $\hat\gamma\colon[0,1]\times[0,1]\to\mathcal{X}$ such that $\hat\gamma_0=\alpha$, $\hat\gamma_1=\beta$ and $\hat\gamma_t\in\Gamma$ for all $t\in[0,1]$.

From Theorem **??**, we have $|\alpha\,\beta|_\Gamma>0$ for distinct α and β; that is,

❻ $|{*}{*}|_\Gamma$ *is a metric on* Γ.

Further, again from Theorem **??**, we have

❼ *The map*

$$\mathfrak{m}\colon\xi\mapsto\xi(1)\colon\Gamma\to\mathcal{U}$$

is a local isometry. In particular, $\operatorname{curv}\Gamma\leqslant\kappa$.

Let $\alpha\colon[0,1]\to\mathcal{U}$ be a path with $\operatorname{length}\alpha<\varpi^\kappa$ and $\alpha(0)=p$. The homotopy constructed in Corollary **??** can be regarded as a path in Γ, say $\hat\alpha\colon[0,1]\to\to\Gamma$, such that $\hat\alpha(0)=\hat p$ and $\mathfrak{m}\circ\hat\alpha=\alpha$; in particular $\hat\alpha(1)=\alpha$. From Claim **??**, $\operatorname{length}\hat\alpha=\operatorname{length}\alpha$; moreover, it follows that α is a local geodesic path of $\mathcal{U}$ if and only if $\hat\alpha$ is a local geodesic path of Γ.

Further, from Corollary **??**, for any $\xi\in\Gamma$ and path $\hat\alpha\colon[0,1]\to\Gamma$ from $\hat p$ to ξ, we have

$$\begin{aligned}\operatorname{length}\hat\alpha=\operatorname{length}\mathfrak{m}\circ\hat\alpha\geqslant\\ \geqslant\operatorname{length}\xi=\operatorname{length}\hat\xi\end{aligned}$$

where equality holds only if $\hat\alpha$ is a reparametrization of $\hat\xi$. In particular,

❽ $$|\hat p\,\xi|_\Gamma=\operatorname{length}\xi$$

and $\hat{\xi}\colon [0,1] \to \Gamma$ is the unique geodesic path from $\hat{p}$ to ξ. Clearly, the map $\xi \mapsto \hat{\xi}$ is continuous.

From **??** and Proposition **??**, it follows

❾ *For any $\bar{R} < \varpi^\kappa/2$, the closed ball $\overline{\mathrm{B}}[\hat{p}, \bar{R}]$ in Γ is complete.*

Take $\mathcal{B} = \mathrm{B}(\hat{p}, \varpi^\kappa/2)$ and ♍ constructed above. According to the Radial lemma (**??**), $\mathcal{B} \in \mathrm{CAT}\lceil\kappa\rceil$. The remaining statements are already proved. □

1.8 Majorization

1.8.1. Definition. *Let $\mathcal{X}$ be a metric space, $\tilde{\alpha}$ be a simple closed curve of finite length in $\mathbb{M}^2[\kappa]$, and $D \subset \mathbb{M}^2[\kappa]$ be a closed region bounded by $\tilde{\alpha}$. A length-nonincreasing map $F\colon D \to \mathcal{X}$ is called* majorizing *if it is length-preserving on $\tilde{\alpha}$.*

In this case, we say that D majorizes *the curve $\alpha = F \circ \tilde{\alpha}$ under the map F.*

The following proposition is an immediate consequence of the definition.

1.8.2. Proposition. *Let α be a closed curve in a metric space $\mathcal{X}$. Suppose $D \subset \mathbb{M}^2[\kappa]$ majorizes α under $F\colon D \to \mathcal{X}$. Then any geodesic subarc of α is the image under F of a subarc of ∂D that is geodesic in the intrinsic metric of D.*

In particular, if D is convex then the corresponding subarc is a geodesic in $\mathbb{M}^2[\kappa]$.

Proof. For a geodesic subarc $\gamma\colon [a,b] \to \mathcal{X}$ of $\alpha = F\circ\tilde{\alpha}$, write $\tilde{\gamma} = (F|\partial D)^{-1}\circ\gamma$, $s = \mathrm{length}\,\gamma$, $\tilde{s} = \mathrm{length}\,\tilde{\gamma}$, $r = |\gamma(a)\ \gamma(b)|_{\mathcal{X}}$, and $\tilde{r} = |\tilde{\gamma}(a)\ \tilde{\gamma}(b)|_D$. Then

$$\tilde{r} \geqslant r = s = \tilde{s} \geqslant \tilde{r}.$$

Therefore $\tilde{s} = \tilde{r}$. □

1.8.3. Corollary. *Let $[pxy]$ be a triangle of perimeter $< 2\cdot\varpi^\kappa$ in a metric space $\mathcal{X}$. Assume a convex region $D \subset \mathbb{M}^2[\kappa]$ majorizes $[pxy]$. Then $D = \mathrm{Conv}[\tilde{p}\tilde{x}\tilde{y}]$ for a model triangle $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}^\kappa(pxy)$, and the majorizing map sends $\tilde{p}$, $\tilde{x}$ and $\tilde{y}$ respectively to p, x and y.*

Now we come to the main theorem of this section.

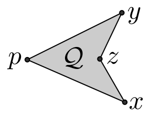
1.8.4. Reshetnyak majorization theorem. *Any closed curve α of length $< 2\cdot\varpi^\kappa$ in $\mathcal{U} \in \mathrm{CAT}\lceil\kappa\rceil$ is majorized by a convex region in $\mathbb{M}^2[\kappa]$.*

This theorem is proved in [**?**]; our proof also uses a trick from [**?**].

The theorem is not trivial even when α is a triangle, say $[pxy]$. In this case, by Proposition **??**, the majorizing convex region has to be isometric to $\mathrm{Conv}[\tilde{p}\tilde{x}\tilde{y}]$, where $[\tilde{p}\tilde{x}\tilde{y}] = \tilde{\triangle}^\kappa(pxy)$.

In our proof of the theorem, we find a majorizing map for $[pxy]$ whose image W is the image of the line-of-sight map (definition **??**) for $[xy]$ from p. The construction of the correct parameterization is given in lemmas **??** and **??**. This map is used to construct a majorizing map for a polygon in $\mathcal{U}$, and passing to a limit we get the general case.

Finding such a parameterization $F\colon \mathrm{Conv}[\tilde p\tilde x\tilde y] \to W$ is not as simple as one might think. In particular, the map $F\colon \mathrm{Conv}[\tilde p\tilde x\tilde y] \to W : \tilde\gamma_t(s) \mapsto \gamma_t(s)$ that sends the point with parameter (t,s) under the line-of-sight map for $[\tilde x\tilde y]$ from $\tilde p$, to the point with the same parameter under the line-of-sight map for $[xy]$ from p, is not majorizing in general. The following example illustrates this point



Example. Let $\mathcal{Q}$ be a bounded region in $\mathbb{E}^2$ formed by two congruent triangles, where the bounding quadrangle $[pxzy]$ is nonconvex at z (as in the picture). Equip $\mathcal{Q}$ with the intrinsic metric. Then $\mathcal{Q} \in \overline{\mathrm{CAT}}[0]$ by Reshetnyak gluing (**??**). For triangle $[pxy]_{\mathcal{Q}}$ in $\mathcal{Q}$ and its model triangle $[\tilde p\tilde x\tilde y]$ in $\mathbb{E}^2$, we have $|\tilde x\tilde y| = |xy|_{\mathcal{Q}} = |xz| + |zy|$. Then the map F defined by matching line-of-sight parameters satisfies $F(\tilde x) = x$ and $|xF(\tilde w)| > |\tilde x\tilde w|$ if $\tilde w$ is near the midpoint $\tilde z$ of $[\tilde x\tilde y]$ and lies on $[\tilde p\tilde z]$. Indeed, by the first variation formula (**??**), for $\varepsilon = 1 - s$ we have $|\tilde x\tilde w| = |\tilde x\tilde\gamma_{\frac12}(s)| = |xz| + o(\varepsilon)$ and $|xF(\tilde w)| = |x\gamma_{\frac12}(s)| = |xz| - \varepsilon\cdot\cos\measuredangle[z\,{}^{p}_{x}] + o(\varepsilon)$. Thus F is not majorizing.

In the following proofs, $x^1\dots x^n$ $(n \geqslant 3)$ denotes a broken geodesic with vertices $x^1,\dots,x^n$, and $[x^1\dots x^n]$ denotes the corresponding (closed) polygon. For a subset R of the ambient metric space, we denote by $[x^0\dots x^n]_R$ a polygon in the length metric of R.

Our first lemma gives a model space construction based on repeated application of Lemma **??**.

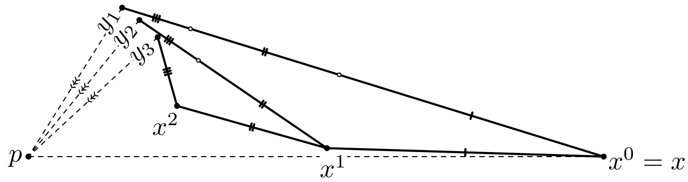
1.8.5. Lemma. *In $\mathbb{M}^2[\kappa]$, let β be a curve from x to y which is concave with respect to p. Let D be the subgraph of β with respect to p. Assume*

$$\operatorname{length}\beta + |px| + |py| < 2\cdot\varpi^\kappa.$$

a) Then β forms a geodesic $[xy]_D$ in D and therefore β, $[px]$ and $[py]$ form a triangle $[pxy]_D$ in the intrinsic metric of D.
b) Let $[\tilde p\tilde x\tilde y]$ be the model triangle for triangle $[pxy]_D$. Then there is a short map $G\colon \mathrm{Conv}[\tilde p\tilde x\tilde y] \to D$ such that $\tilde p \mapsto p$, $\tilde x \mapsto x$, $\tilde y \mapsto y$, and G is length-preserving on each side of $[\tilde p\tilde x\tilde y]$. In particular, $\mathrm{Conv}[\tilde p\tilde x\tilde y]$ majorizes triangle $[pxy]_D$ in D under G.

Proof. We prove the lemma for a broken geodesic β; the general case then follows by approximation. Namely, since β is concave, it can be approximated by broken geodesics that are concave with respect to p, with their lengths converging to $\operatorname{length}\beta$. Applying Lemma **??**, we can pass to a limit to obtain the needed map G.

Suppose $\beta = x^0x^1\dots x^n$ is polygonal curve with $x^0 = x$ and $x^n = y$. Consider a sequence of broken geodesics $\beta_i = x^0x^1\dots x^{i-1}y_i$ such that $|py_i| = |py|$ and β_i has same length as β; that is, $|x^{i-1}\,y_i| = |x^{i-1}\,x^i| + |x^i\,x^{i+1}| + \dots$ $\dots + |x^{n-1}\,x^n|$.



Clearly $\beta_n = \beta$. Sequentially applying Alexandrov's lemma (**??**) shows that each of the broken geodesics $\beta_{n-1}, \beta_{n-2}, \ldots, \beta_1$ is concave with respect to p. Let D_i to be subgraph of β_i with respect to p. Applying Lemma **??** gives a short map $G_i\colon D_i \to D_{i+1}$ that maps $y_i \mapsto y_{i+1}$ and does not move p and x (in fact, G_i is the identity everywhere except on $\mathrm{Conv}[px^{i-1}y_i]$). Thus the composition $G_{n-1} \circ \ldots \circ G_1\colon D_1 \to D_n$ is a short map.

The result follows since $D_1 \overset{iso}{=\!=} \mathrm{Conv}[\tilde p\tilde x\tilde y]$. □

1.8.6. Lemma. *In $\mathcal{U} \in \mathrm{CAT}[\kappa]$, let $[pxy]$ be a triangle in $\mathcal{U}$ of perimeter $< 2\cdot\varpi^\kappa$. In $\mathbb{M}^2[\kappa]$, let $\tilde\gamma$ be the κ-development of $[xy]$ with respect to p, where $\tilde\gamma$ has basepoint $\tilde p$ and subgraph D. Consider the map $H\colon D \to \mathcal{U}$ that sends the point with parameter (t, s) under the line-of-sight map for $\tilde\gamma$ from $\tilde p$, to the point with the same parameter under the line-of-sight map f for $[xy]$ from p. Then H is length-nonincreasing. In particular, D majorizes triangle $[pxy]$.*

Proof. Let $\gamma = \mathrm{geod}_{[xy]}$ and $T = |xy|$. As in the proof of the development criterion (**??**), take a partition $0 = t^0 < t^1 < \ldots < t^n = T$, and set $x^i = \gamma(t^i)$. Construct a chain of model triangles $[\tilde p\tilde x^{i-1}\tilde x^i] = \tilde\triangle^\kappa(px^{i-1}x^i)$, with $\tilde x^0 = \tilde x$ and the direction of $[\tilde p\tilde x^i]$ turning counterclockwise as i grows. Let D_n be the subgraph with respect to $\tilde p$ of the broken geodesic $\tilde x^0 \ldots \tilde x^n$.

Let δ_n be the maximum radius of a circle inscribed in any of the triangles $[\tilde p\tilde x^{i-1}\tilde x^i]$.

Now we construct a map $H_n\colon D_n \to \mathcal{U}$ that increases distances by at most $2\cdot\delta_n$.

Suppose $x \in D_n$. Then x lies on or inside some triangle $[\tilde p\tilde x^{i-1}\tilde x^i]$. Define $H_n(x)$ by first mapping x to a nearest point on $[\tilde p\tilde x^{i-1}\tilde x^i]$ (choosing one if there are several), followed by the natural map to the triangle $[px^{i-1}x^i]$.

Since triangles in $\mathcal{U}$ are κ-thin (**??**), the restriction of H_n to each triangle $[\tilde p\tilde x^{i-1}\tilde x^i]$ is short. Then the triangle inequality implies that the restriction of H_n to

$$U_n = \bigcup_{1\leqslant i\leqslant n} [\tilde p\tilde x^{i-1}\tilde x^i]$$

is short with respect to the intrinsic metric on D_n. Since nearest-point projection from D_n to U_n increases the D_n-distance between two points by at most $2\cdot\delta_n$, the map H_n also increases the D_n-distance by at most $2\cdot\delta_n$.

Consider $y_n \in D_n$ with $y_n \to y \in D$ and $z_n \in D_n$ with $z_n \to z \in D$. Since $\delta_n \to 0$ under increasingly finer partitions, and geodesics in $\mathcal{U}$ vary continuously with their endpoints (**??**), we have $H_n(x_n) \to H(x)$ and $H_n(y_n) \to H(y)$. Since

$$|H_n(x_n)\ H_n(y_n)| \leqslant |x_n\, y_n|_{D_n} + 2\cdot\delta_n,$$

where the lefthand side converges to $|H(x)\ H(y)|$ and the righthand side converges to $|x\, y|_D$, it follows that H is short. □

*Proof of Reshetnyak majorization theorem (**??**).* We begin by proving the theorem in case α is polygonal.

First suppose α is a triangle, say $[pxy]$. By assumption, the perimeter of $[pxy]$ is $< 2\cdot\varpi^\kappa$.

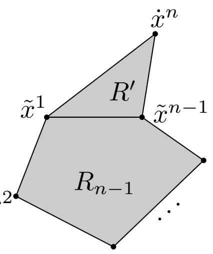
Let $\tilde\gamma$ be the κ-development of $[xy]$ with respect to p, where $\tilde\gamma$ has basepoint $\tilde p$ and subgraph D. By the development criterion (**??**), $\tilde\gamma$ is concave.

By Lemma **??**, there is a short map $G\colon \operatorname{Conv}\tilde\triangle^\kappa(pxy) \to D$. Further, by Lemma **??**, D majorizes $[pxy]$ under a majorizing map $H\colon D \to \mathcal{U}$. Clearly $H \circ G$ is a majorizing map for $[pxy]$.

Now we claim that any closed n-gon $[x^1x^2\dots x^n]$ of length $< 2\cdot\varpi^\kappa$ in a $\overline{\mathrm{CAT}}[\kappa]$ space is majorized by a convex polygonal region $R_n = \operatorname{Conv}[\tilde x^1\tilde x^2\dots$ $\dots\tilde x^n]$ in $\mathbb{M}^2[\kappa]$, under a map F_n such that $F_n\colon \tilde x^i \mapsto x^i$ for each i.

Assume the statement is true for $(n-1)$-gons, $n \geqslant 4$. Then $[x^1x^2\dots x^{n-1}]$ is majorized by a convex polygonal region

$$R_{n-1} = \operatorname{Conv}(\tilde x^1, \tilde x^2, \dots, \tilde x^{n-1}),$$



in $\mathbb{M}^2[\kappa]$ under a map F_{n-1} satisfying $F_{n-1}(\tilde x^i) = x^i$ for all i. Take $\dot x^n \in \mathbb{M}^2[\kappa]$ such that $[\tilde x^1\tilde x^{n-1}\dot x^n] = \tilde\triangle^\kappa(x^1x^{n-1}x^n)$ and this triangle lies on the other side of $[\tilde x^1\tilde x^{n-1}]$ from R_{n-1}. Let $\dot R = \operatorname{Conv}[\tilde x^1\tilde x^{n-1}\dot x^n]$, and $\dot F\colon \dot R \to \mathcal{U}$ be a majorizing map for $[x^1x^{n-1}x^n]$ as provided above.

Set $R = R_{n-1} \cup \dot R$, where R carries its intrinsic metric. Since F_n and F agree on $[\tilde x^1\tilde x^{n-1}]$, we may define $F\colon R \to \mathcal{U}$ by

$$F(x) = \begin{cases} F_{n-1}(x), & x \in R_{n-1},\\ \dot F(x), & x \in \dot R. \end{cases}$$

Then F is length-nonincreasing, and is a majorizing map for $[x^1x^2\dots x^n]$ (as in Definition **??**).

If R is a convex subset of $\mathbb{M}^2[\kappa]$, we are done.

If R is not convex, the total internal angle of R at $\tilde x^1$ or $\tilde x^{n-1}$ or both is $> \pi$. By relabeling we may suppose the $\measuredangle\tilde x^{n-1} > \pi$. The region R is is obtained by glueing R_{n-1} to $\dot R$ by $[x^1x^{n-1}]$. Thus, by Reshetnyak gluing (**??**), R in its intrinsic metric is a $\overline{\mathrm{CAT}}[\kappa]$-space. Moreover $[\tilde x^{n-2}\tilde x^{n-1}] \cup [\tilde x^{n-1}\dot x^n]$ is a geodesic of R. Thus $[\tilde x^1, \tilde x^2, \dots, \tilde x^{n-1}\dot x^n]_R$ is a closed $(n-1)$-gon in R, to which the induction hypothesis applies. The resulting short map from a convex region in $\mathbb{M}^2[\kappa]$ to R, followed by F, is the desired majorizing map.

Note that in fact we proved the following:

❶ *Let F_{n-1} be a majorizing map for the polygon $[x^1x^2\dots x^{n-1}]$, and $\dot F$ be a majorizing map for the triangle $[x^1x^{n-1}x^n]$. Then there is a majorizing map F_n for the polygon $[x^1x^2\dots x^n]$ such that*

$$\operatorname{Im} F_{n+1} = \operatorname{Im} F_n \cup \operatorname{Im} \dot F.$$

We now use this claim to prove the theorem for general curves.

Assume $\alpha\colon [0,\ell] \to \mathcal{U}$ is an arbitrary closed curve with natural parameter. Choose a sequence of partitions $0 = t_n^0 < t_n^2 < \dots < t_n^n = \ell$ so that:

- ⋄ The set $\{t_{n+1}^i\}_{i=0}^{n+1}$ is obtained from the set $\{t_n^i\}_{i=0}^n$ by adding one element.
- ⋄ For some sequence $\varepsilon_n \to 0+$, we have $t_n^i - t_n^{i-1} < \varepsilon_n$ for all i.

Inscribe in α a sequence of polygons P_n with vertices $\alpha(t_n^i)$. Apply the claim above, to get a sequence of majorizing maps $F_n\colon R_n \to \mathcal{U}$. Note that for all $m > n$

- ⋄ $\operatorname{Im} F_m$ lies in an ε_n-neighborhood of $\operatorname{Im} F_n$
- ⋄ $\operatorname{Im} F_m \setminus \operatorname{Im} F_n$ lies in an ε_n-neighborhood of α.

It follows that the set

$$K = \alpha \cup \left(\bigcup_n \operatorname{Im} F_n \right)$$

is compact. Therefore the sequence (F_n) has a partial limit as $n \to \infty$; say F. Clearly F is a majorizing map for α. ☐

1.9 Angles

Recall that ω denotes a nonpriciple ultrafilter, see Section **??**.

1.9.1. Angle semicontinuity. *Let $\mathcal{U}_n \in \mathrm{CAT}[\kappa]$ and $\mathcal{U}_n \to \mathcal{U}_\omega$ as $n \to \omega$. Assume that a sequence of hinges $[p_n\,{}^{x_n}_{y_n}]$ in $\mathcal{U}_n$ converges to a hinge $[p_\omega\,{}^{x_\omega}_{y_\omega}]$ in $\mathcal{U}_\omega$. Then*

$$\measuredangle[p_\omega\,{}^{x_\omega}_{y_\omega}] \geqslant \lim_{n\to\omega} \measuredangle[p_n\,{}^{x_n}_{y_n}].$$

Proof. By angle-sidelength monotonicity (**??**),

$$\measuredangle[p_\omega\,{}^{x_\omega}_{y_\omega}] = \inf\left\{\, \tilde\measuredangle^\kappa(p_\omega\,{}^{\bar x_\omega}_{\bar y_\omega}) \;\middle|\; \bar x_\omega \in\,]p_\omega x_\omega]\,,\ \bar y_\omega \in\,]p_\omega y_\omega] \,\right\}.$$

For fixed $\bar x_\omega \in\,]p_\omega x_\omega]$ and $\bar y_\omega \in\,]p_\omega x_\omega]$, choose $\bar x_n \in\,]p x_n]$ and $\bar y_n \in\,]p y_n]$ so that $\bar x_n \to \bar x_\omega$ and $\bar y_n \to \bar y_\omega$ as $n \to \omega$. Clearly

$$\tilde\measuredangle^\kappa(p_n\,{}^{\bar x_n}_{\bar y_n}) \to \tilde\measuredangle^\kappa(p_\omega\,{}^{\bar x_\omega}_{\bar y_\omega})$$

as $n \to \omega$.

By angle comparison (**??**), $\measuredangle[p_n\,{}^{x_n}_{y_n}] \leqslant \tilde\measuredangle^\kappa(p_n\,{}^{\bar x_n}_{\bar y_n})$. Hence the result. ☐

Now we verify that the first variation formula holds in the CAT setting. Compare first variation inequality (**??**) for general metric spaces and the Strong angle lemma (**??**) for $\overline{\mathrm{CBB}}$ spaces.

1.9.2. Strong angle lemma. *Let $\mathcal{U} \in \mathrm{CAT}$. Then for any hinge $[p\,{}^q_x]$ in $\mathcal{U}$, we have*

$$\measuredangle[p\,{}^q_x] = \lim_{\substack{\bar x\to p\\ \bar x\in]px]}} \tilde\measuredangle^\kappa(p\,{}^q_{\bar x}) \tag{❶}$$

for any $\kappa \in \mathbb{R}$ such that $|pq| < \varpi^\kappa$.

Proof. By angle-sidelength monotonicity (**??**), the righthand side is defined and at least equal to the lefthand side.

By Lemma **??**, we may take $\kappa = 0$ in **??**. By the cosine law and the first variation inequality (**??**), the righthand side is at most equal to the lefthand side. ☐

1.9.3. First variation formula. *Let $\mathcal{U} \in \mathrm{CAT}[\kappa]$. For any geodesic $[py]$ in $\mathcal{U}$ and point q such that $|pq| < \varpi^\kappa$, we have*

$$|q\,\mathrm{geod}_{[py]}(t)| = |qp| - t\cdot \cos\measuredangle[p\,{}^q_y] + o(t).$$

Proof. The first variation formula is equivalent to the strong angle lemma (**??**), as follows from the Euclidean cosine law. ☐

The No-conjugate-point theorem **??** immediately gives an extension of the first variation formula (**??**) from geodesics in CAT$\lceil\kappa\rceil$ spaces to local geodesics in spaces of curvature $\leqslant \kappa$.

1.9.4. Corollary. *Suppose $\mathcal{U}$ is a locally complete metric space with* $\operatorname{curv}\mathcal{U} \leqslant \kappa$. *For any geodesic $[px]$ in $\mathcal{U}$ and local geodesic $\gamma\colon [0,1] \to \mathcal{U}$ from p to q of length $< \varpi^\kappa$, let $\gamma_t\colon [0,1] \to \mathcal{U}$ be a continuous family of local geodesic paths with $\gamma_0 = \gamma$, $\gamma_t(0) = \operatorname{geod}_{[px]}(t)$ and $\gamma_t(1) = q$. Then*

$$\operatorname{length}(\gamma_t) = \operatorname{length}(\gamma) - t\cdot\cos\measuredangle(\gamma,[px]) + o(t).$$

The following lemma was proved in [**?**]; it states that in a CAT$\lceil\kappa\rceil$ space, a sharp triangle comparison implies the presence of an isometric copy of the convex hull of the model triangle. The analogue for $\overline{\mathrm{CBB}}\lfloor\kappa\rfloor$-spaces fails (see Exercise **??**).

1.9.5. Flat triangles lemma. *Suppose $\mathcal{U} \in$ CAT$\lceil\kappa\rceil$. For a triangle $[xyz]$ in $\mathcal{U}$ with model triangle $[\tilde x\tilde y\tilde z] = \tilde\triangle_\kappa(xyz)$, the following are equivalent:*

a) $\measuredangle[x\,{}^y_z] = \tilde\measuredangle^\kappa(x\,{}^y_z)$;

b) $|xw| = |\tilde x\tilde w|$ for some $\tilde w \in]\tilde x\tilde z[$, where $w = f(\tilde w)$ and f is the natural map;

c) there is a distance-preserving map $F\colon \operatorname{Conv}[\tilde x\tilde y\tilde z] \to \mathcal{U}$ that maps $\tilde x, \tilde y, \tilde z$ to x, y, z respectively.

Proof. By angle-sidelength monotonicity (**??**),

❷ *If $\measuredangle[x\,{}^y_z] = \tilde\measuredangle^\kappa(x\,{}^y_z)$, then the restriction $f|([xy]\cup[xz])$ is distance-preserving.*

Suppose $v, w \in]xy[$, in the order x, v, w, y. Applying **??** to triangles $[zxy]$, $[zxw]$ and $[zvy]$, we find that $|zv| = |\tilde z\tilde v|$ if and only if $|zw| = |\tilde z\tilde w|$, where $v = f(\tilde v)$, $w = f(\tilde w)$. Thus by the first variation formula, we have

❸ *If $|zw| = |\tilde z\tilde w|$ for some $w \in]xy[$, then $\measuredangle[y\,{}^x_z] = \tilde\measuredangle^\kappa(y\,{}^x_z)$.*

*(**??**)⇔(**??**).* Apply claims **??** and **??** for permutations of x, y, z.

*(**??**)⇔(**??**).* Note that (**??**) implies that the natural map $f\colon [\tilde x\tilde y\tilde z] \to [xyz]$ is distance-preserving. Then by the triangle inequality for angles at x,

❹ $$\measuredangle[x\,{}^v_w] = \tilde\measuredangle^\kappa(x\,{}^v_w)$$

for any $v, w \in]yz[$.

Let $(t,s) \mapsto \gamma_t(s)$ be the line-of-sight map for $[yz]$ from x (defined in **??**), and $(t,s) \mapsto \tilde\gamma_t(s)$ be the line-of-sight map for $[\tilde y\tilde z]$ from $\tilde x$. By **??** and angle-sidelength monotonicity (**??**), the map $F\colon \operatorname{Conv}[\tilde x\tilde y\tilde z] \to \mathcal{U}$ defined by $F\colon \tilde\gamma_t(s) \mapsto \gamma_t(s)$ is distance-preserving. Hence (**??**).

The implication (**??**)⇒(**??**) holds trivially. □

1.10 Comments and open problems

The following question was known in folklore in the 80's, but it seems that in print it was first mentioned in [**?**]. We do not see any reason why it should be true, but we also cannot construct a counterexample.

1.10.1. Open question. *Let* $\mathcal{U} \in \overline{\mathrm{CAT}}[0]$ *and* $K \subset \mathcal{U}$ *be a compact set. Is it true that* K *lies in a convex compact set* $\bar{K} \subset \mathcal{U}$*?*

The question can easily be reduced to the case when K is finite; so far it is not even known if any three points in a CAT[0]-space lie in a compact convex set.

1.11 Exercises

1.11.1. Exercise. *Show that the first variation formula holds when the endpoint follows any curve with a direction, not necessarily a geodesic:*

Let $\mathcal{U} \in \mathrm{CAT}[\kappa]$*, and* $\gamma : [0, \varepsilon] \to \mathcal{U}$ *be a curve for which* $\gamma(0) = p$ *and* $\gamma^+(0)$ *exists. For any point* q *such that* $|pq| < \varpi^\kappa$*, let* α *be the angle at* p *between* $\gamma^+(0)$ *and* $[qp]$*. Then*

$$|q\gamma(t)| = |qp| - t\cdot|\gamma^+(0)|\cdot\cos\alpha + o(t).$$

1.11.2. Exercise. *A* convex space $\mathcal{X}$ *is a geodesic space such that the function* $t \mapsto |\gamma(t)\ \sigma(t)|$ *is convex for any two geodesics* $\gamma, \sigma : [0, a) \to \mathcal{X}$*. A* locally convex space *is an intrinsic space in which every point has a neighborhood that is a convex space in the restricted metric. Show:*

a) If $\mathcal{U} \in \mathrm{CAT}[0]$*, then* $\mathcal{U}$ *is a convex space.*

b) (Hadamard-Cartan Theorem for locally convex spaces.) A complete, simply connected, locally convex space is a convex space.

1.11.3. Exercise. *Suppose* $\mathcal{U}$ *is a complete intrinsic space with* $\operatorname{curv}\mathcal{U} \leqslant 1$*, and the space of all closed curves in* $\mathcal{U}$ *with length* $< 2\cdot\pi$ *is connected. Prove that* $\mathcal{U} \in \overline{\mathrm{CAT}}[1]$*.*

The following exercise is the rigidity case of the Majorization theorem.

1.11.4. Exercise. *Let* $\mathcal{U} \in \mathrm{CAT}[\kappa]$ *and* $\alpha\colon [0, \ell] \to \mathcal{U}$ *be a closed curve with arclength parametrization. Assume that* $\ell < 2\cdot\varpi^\kappa$ *and there is a closed convex curve* $\tilde{\alpha}\colon [0, \ell] \to \mathbb{M}^2[\kappa]$ *such that*

$$|\alpha(t_0)\ \alpha(t_1)|_{\mathcal{U}} = |\tilde{\alpha}(t_0)\ \tilde{\alpha}(t_1)|_{\mathbb{M}^2[\kappa]}$$

for any t_0 *and* t_1*. Then there is an isometric embedding* $F\colon \operatorname{Conv}\tilde{\alpha} \to \mathcal{U}$ *such that* $F\colon \tilde{\alpha}(t) \mapsto \alpha(t)$ *for any* t*.*

Hints. *(Easier way.)* Let $(t, s) \mapsto \gamma_t(s)$ be the line-of-sight map for α from $\alpha(0)$, and $(t, s) \mapsto \tilde{\gamma}_t(s)$ be the line-of-sight map for $\tilde{\alpha}$ from $\tilde{\alpha}(0)$. Consider the map $F\colon \operatorname{Conv}\tilde{\alpha} \to \mathcal{U}$ such that $F\colon \tilde{\gamma}_t(s) \mapsto \gamma_t(s)$.

Show that F majorizes α and conclude that F is distance-preserving.

(Harder way.) Prove and apply the following lemma together with the Majorization theorem.

1.11.5. Lemma. *Let* α *and* β *be two convex curves in* $\mathbb{M}^2[\kappa]$*. Assume*

$$\operatorname{length}\alpha = \operatorname{length}\beta < 2\cdot\varpi^\kappa$$

and there is a short bijecction $f\colon \alpha \to \beta$. Then f is an isometry.

1.11.6. Exercise. *In $\mathcal{U} \in \mathrm{CAT}[\kappa]$, show that if a closed curve α of length $< 2\cdot\varpi^\kappa$ is not a triangle, then exactly one of these statements holds:*

a) *α is majorized by two non-isometric convex regions in $\mathbb{M}^2[\kappa]$;*

b) *the majorizing map for α is distance preserving.*