BOARD NOTES ON ALEXANDROV GEOMETRY

- *Summer School "Metric Geometry", 25-30 August 2013, Les Diablerets, Switzerland*
- *Summer School "Geometric Analysis on Riemannian and Singular Metric Spaces", 30 Sept - 4 Oct 2013, Como, Italy*

Our minicourse will consider some of the topics included in these notes, as well as other material. The notes will be used as a reference.

There are many important omissions of theory in the notes, but it is hoped they will allow easier access to more advanced topics. We also hope to give a hint of the beauty and power of the subject.

The notes draw on our book [AKP].

Other general sources for this subject include: [BB I01] ; [BH], [BS 07] for curvature bounded above ; [BGP 92], [S 93], [Pl 02] for curvature bounded below.

Special thanks to Anton Petrunin for many interesting discussions and ideas.

Contents

1. Model space

(1) (a) $\mathcal{M}(\kappa) = \mathbf{E}^2,\ \mathbf{S}^2(\kappa),\ \mathbf{H}^2(\kappa)$ (for $\kappa = 0$, $\kappa > 0$, $\kappa < 0$ respectively).

(b) Set $\pi^\kappa = \pi/\sqrt{\kappa}$ $(= \infty$ if $\kappa \le 0)$.

Exercise 1.1. Show a closed curve of length $< 2 \cdot \pi^\kappa$ in $\mathbf{S}^2(\kappa)$ lies in an open hemisphere.

(2) Let $\big(\mathcal{X}\,,\ |{*}{*}| : \mathcal{X} \times \mathcal{X} \to [0, \infty]\big)$ be a metric space.

(a) A *geodesic* γ joining x^1 and x^2 is a constant-speed path of length $|x^1x^2|$. For a *local geodesic* γ, the parameter interval is covered by open subintervals on which γ is geodesic. We write $[x^1x^2]$ for either γ or its image. *The choice may not be unique, but once we write $[xy]$ it means we have made a choice.*

(b) $\mathcal{X}$ is *geodesic* (*intrinsic*) if any $x^2, x^2 \in \mathcal{X}$ are joined by a geodesic (respectively, by curves of length arbitrarily close to $|x^1x^2|$). For *r-geodesic* (*r-intrinsic*), take $|x^1\,x^2| < r$.

(c) A *triangle* $[x^1x^2x^3]$ in $\mathcal{X}$ is $[x^1x^2] \cup [x^2x^3] \cup [x^3x^1]$.

(d) For $x^1, x^2, x^3 \in \mathcal{X}$, the *model triangle* $\tilde{\triangle}^\kappa[x^1\,x^2\,x^3]$ in $\mathcal{M}(\kappa)$ is the triangle with sidelengths $|x^1\,x^2|$, $|x^2\,x^3|$, $|x^3\,x^1|$. The model angle $\tilde{\measuredangle}^\kappa\big[x^1\,{}^{x^2}_{x^3}\big]$ is the angle corresponding to x^1 in $\tilde{\triangle}^\kappa\big[x^1\,x^2\,x^3\big]$. These are said to be *defined* if the sum of sidelengths is $< 2 \cdot \pi^\kappa$.

(3) (a) For a *hinge* $\big[p\,{}^x_y\big] = [px] \cup [py]$ in $\mathcal{X}$, define the angle $\measuredangle\big[p\,{}^x_y\big]$ by

$$\measuredangle\big[p\,{}^x_y\big] = \lim_{\bar{x}, \bar{y} \to p} \tilde{\measuredangle}^\kappa\big[p\,{}^{\bar{x}}_{\bar{y}}\big],$$

where $\bar{x} \in [px], \bar{y} \in [py]$, if the limit exists.

(b) This definition of angles in $\mathcal{X}$ is independent of κ, e.g.

$$\tilde{\measuredangle}^\kappa\big[p\,{}^x_y\big] - \tilde{\measuredangle}^{\bar{\kappa}}\big[p\,{}^x_y\big] \ \le\ C(\kappa, \bar{\kappa}) \cdot |px| \cdot |py|.$$

(c) *Triangle inequality for angles* holds:

$$\measuredangle\big[p\,{}^x_z\big] \le \measuredangle\big[p\,{}^x_y\big] + \measuredangle\big[p\,{}^y_z\big],$$

if all three angles are defined. (Use $\kappa = 0$, Euclidean cosine formula, triangle inequality for distance.)

2. Definitions

• In this column, assume $\mathcal{X}$ complete, π^κ-geodesic.

(1) (a) We say $\mathcal{X} \in \mathrm{CAT}^\kappa \;\Leftrightarrow$

$\boxed{\text{Point-Side}}$ holds:

if $\tilde\triangle^\kappa[pxy] = [\tilde p\tilde x\tilde y]$ is defined, $z \in [xy]$, $\tilde z \in [\tilde x\tilde y]$, $|xz| = |\tilde x\tilde z|$, then

$$(*) \quad |pz| \;\le\; |\tilde p\tilde z|.$$

(b) $(*) \;\Leftrightarrow\; \tilde\angle^\kappa\big[x\,{}^{p}_{z}\big] \le \tilde\angle^\kappa\big[x\,{}^{p}_{y}\big]$.

(c) *Examples*: $\mathcal{X} =$
Tree; locally-$\mathbf{E}^2$ sector with any angle; locally-$\mathbf{E}^2$ cone with angle $\ge 2\pi$.

(d) Geodesics can bifurcate.

(e) Geodesics (length $< \pi\kappa$) are unique.

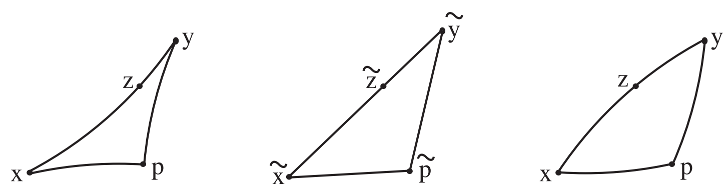
• In this column, assume $\mathcal{X}$ complete, intrinsic, $\dim_{\mathrm{Haus}} \mathcal{X} < \infty$.[1]

(1) (a) We say $\mathcal{X} \in \mathrm{CBB}^\kappa \;\Leftrightarrow$

$\boxed{\text{Point-Side}}$ holds:

if $\tilde\triangle^\kappa[pxy] = [\tilde p\tilde x\tilde y]$ is defined, $z \in [xy]$, $\tilde z \in [\tilde x\tilde y]$, $|xz| = |\tilde x\tilde z|$, then

$$(*) \quad |pz| \;\ge\; |\tilde p\tilde z|.$$



(b) $(*) \;\Leftrightarrow\; \tilde\angle^\kappa\big[x\,{}^{p}_{z}\big] \ge \tilde\angle^\kappa\big[x\,{}^{p}_{y}\big]$.

(c) *Examples*: $\mathcal{X} =$
Closed convex subset of $\mathbf{E}^n$; locally-$\mathbf{E}^2$ cone with angle $\le 2\pi$.

(d) Geodesics cannot bifurcate.

(e) Geodesics need not be unique.

(2) (a) By monotonicity in (1) (b), for a hinge $[px] \cup [py]$:

$\boxed{\text{Point-Side}} \;\Rightarrow\; \angle\big[p\,{}^{x}_{y}\big]$ exists.

$\boxed{\text{Point-Side}} \;\Rightarrow\; \angle\big[p\,{}^{x}_{y}\big]$ exists.

(b) Also by (1) (b), $\boxed{\text{Point-Side}} \;\Rightarrow$ the following $\boxed{\text{Angle}}$ comparisons:

$\angle\big[x\,{}^{p}_{y}\big] \le \tilde\angle^{\bar\kappa}\big[x\,{}^{p}_{y}\big]$.

$\angle\big[x\,{}^{p}_{y}\big] \ge \tilde\angle^{\bar\kappa}\big[x\,{}^{p}_{y}\big]$.

Exercise 2.1. *Semicontinuity of angles.* If $\big[p_n\,{}^{x_n}_{y_n}\big] \to \big[p\,{}^{x}_{y}\big]$ in $\mathcal{X}$, then

$\mathcal{X} \in \mathrm{CAT}^\kappa \Rightarrow$
$$\angle\big[p\,{}^{x}_{y}\big] \ge \limsup_{n\to\infty} \angle\big[p_n\,{}^{x_n}_{y_n}\big].$$

$\mathcal{X} \in \mathrm{CBB}^\kappa \Rightarrow$
$$\angle\big[p\,{}^{x}_{y}\big] \le \liminf_{n\to\infty} \angle\big[p_n\,{}^{x_n}_{y_n}\big].$$

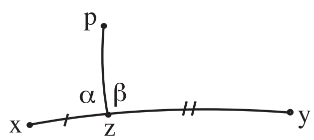
Prove and give examples.

Exercise 2.2. In $\mathcal{X} \in \mathrm{CAT}^\kappa$, show a local geodesic of length $\le \pi^\kappa$ is a geodesic.

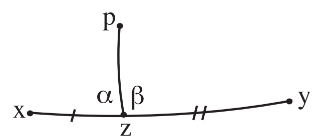
[1] This assumption implies $\mathcal{X}$ is proper, hence π^κ-geodesic. Infinite-dimensional theory is carried out in [AKP].

(3) Proof that $\boxed{\text{Angle}}$ $\Rightarrow$ $\boxed{\text{Point-Side}}$:

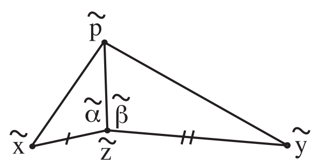
(a) $\mathcal{X} \in \mathrm{CAT}^{\kappa}$:



(a) $\mathcal{X} \in \mathrm{CBB}^{\kappa}$:

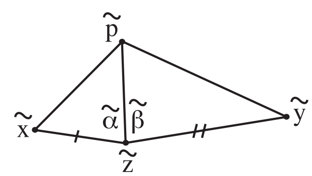


(b) Side-by-side model triangles:



$\tilde\alpha + \tilde\beta \geq \alpha + \beta \geq \pi$ (by (2) (b) & triangle inequality for angles).

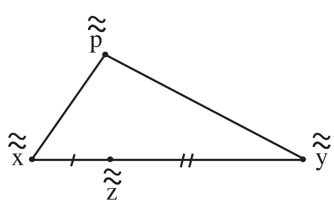
(b) Side-by-side model triangles:



$\tilde\alpha + \tilde\beta \leq \alpha + \beta \leq$??? π ??? (by (2) (b) & ???).

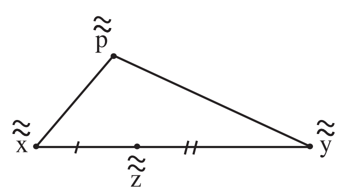
(c) The triangle inequality for angles never switches sign! So in CBB^{κ} column of (2) (b), add to $\boxed{\text{Angle}}$: $(**)$ $\alpha+\beta = \pi$. (You can check that $\boxed{\text{Point-Side}}$ $\Rightarrow (**)$.)

(d) Apply *Alexandrov Lemma* in $\mathcal{M}(\kappa)$ (think of 4 hinged rods in the plane):



"Snap out" to $\tilde\triangle^{\kappa}[p\,x\,y]$:

$|pz| = |\tilde p \tilde z| \leq |\tilde{\tilde p}\tilde{\tilde z}|$.



"Snap in" to $\tilde\triangle^{\kappa}[p\,x\,y]$:

$|pz| = |\tilde p \tilde z| \geq |\tilde{\tilde p}\tilde{\tilde z}|$. $\square$

(e) *Remark:* For both "snap out" and "snap in",

$$\tilde\angle^{\kappa}\left[p\,{}^{x}_{z}\right] + \tilde\angle^{\kappa}\left[p\,{}^{z}_{y}\right] \leq \tilde\angle^{\kappa}\left[p\,{}^{x}_{y}\right].$$

Exercise 2.3. Referring to the figures in (3) (d), (b), (a):

(i) For $\mathcal{X} \in \mathrm{CAT}^\kappa$, find a short, i.e. 1-Lipschitz, map

$$\mathrm{ConvexHull}\,[\tilde{\tilde{p}}\tilde{\tilde{x}}\tilde{\tilde{y}}] \to \ \mathcal{M}(\kappa)$$

that maps $\tilde{\tilde{p}}, \tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{z}}$ to $\tilde{p}, \tilde{x}, \tilde{y}, \tilde{z}$.

(ii) We want a short map

$$\mathrm{ConvexHull}\,[\tilde{\tilde{p}}\tilde{\tilde{x}}\tilde{\tilde{y}}] \ \to \ \mathcal{X}$$

that maps $\tilde{\tilde{p}}, \tilde{\tilde{x}}, \tilde{\tilde{y}}$ to p, x, y (hence maps $[\tilde{\tilde{p}}\tilde{\tilde{x}}\tilde{\tilde{y}}]$ to $[pxy]$). Check that the simplest map you might try is not short, i.e. the *line-of-sight* map taking geodesics that join $\tilde{\tilde{p}}$ to points of $[\tilde{\tilde{x}}\,\tilde{\tilde{y}}]$ proportionally to geodesics that join p to corresponding points of $[x\,y]$.

(iii) Outline of how to get the short map in (ii) (not part of the exercise): Extend your construction in (i) to partition points $\tilde{z}^1, \ldots, \tilde{z}^n$. In limit, get short map

$$\mathrm{ConvexHull}\,[\tilde{\tilde{p}}\tilde{\tilde{x}}\tilde{\tilde{y}}] \to \ \mathcal{M}(\kappa)$$

mapping $\tilde{\tilde{p}}, \tilde{\tilde{x}}, \tilde{\tilde{y}}$ to $\tilde{p}, \tilde{x}, \tilde{y}$, and ConvexHull $[\tilde{\tilde{p}}\tilde{\tilde{x}}\tilde{\tilde{y}}]$ to "development of $[xy]$ from p". It is the line-of-sight map from this development to $\mathcal{X}$ that is short.

(4) Exercise 2.3 is special case, where $\alpha =$ triangle, of:

(a) *Reshetnyak majorization theorem* [R 68]. Suppose $\mathcal{X} \in \mathrm{CAT}^\kappa$, and α is a closed curve in $\mathcal{X}$ of length $< 2 \cdot \pi^\kappa$. Then there is a convex set $\mathrm{CS} \subset \mathcal{M}(\kappa)$ bounded by a curve $\tilde{\alpha}$, and a short map F: $\mathrm{CS} \to \mathcal{X}$, such that $F \circ \tilde{\alpha}$ is length-preserving and $F \circ \tilde{\alpha} = \alpha$.

(5) Lang-Schroeder [LS 97] developed the theory of extendibility of short maps

$$f: \ S \subset \mathcal{X}^1 \in \mathrm{CBB}^\kappa \ \to \ \mathcal{X}^2 \in \mathrm{CAT}^\kappa$$

(also see [AKP 11]). e.g. characterizations of CBB^κ, CBB^κ :

(a) Suppose $\mathcal{X}$ complete, intrinsic, $\dim_{\mathrm{Haus}} \mathcal{X} < \infty$:
$\mathcal{X} \in \mathrm{CBB}^\kappa \iff$ for any 3-point set V_3 and 4-point set V_4, $V_3 \subset V_4 \subset \mathcal{X}$, any short map $f : V_3 \to \mathcal{M}(\kappa)$ extends to a short map $F : V_4 \to \mathcal{M}(\kappa)$.

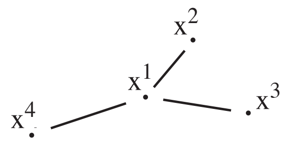
(b) Suppose $\mathcal{X}$ complete, and $\exists$ unique $[xy]$ for any $x, y \in \mathcal{X}$ with $|x\,y| < \pi^\kappa$:
$\mathcal{X} \in \mathrm{CAT}^\kappa \iff$ for any 3-point set V_3 and 4-point set V_4, $V_3 \subset V_4 \subset \mathcal{M}(\kappa)$, where perimeter $V_3 < 2 \cdot \pi^\kappa$, any short map $f : V_3 \to \mathcal{X}$ extends to a short map $F : V_4 \to \mathcal{X}$.

(c) "$\Longleftarrow$" : In both (a) and (b), extendibility condition immediately implies Point-Side.

(6) (a) A quadruple of points x^1, x^2, x^3, x^4 in a metric space $\mathcal{X}$ satisfies $(1+3)$ - *point κ-comparison*, in brief $\boxed{(1+3)^\kappa}$, if

$$\tilde\measuredangle^\kappa\left[x^1\,{x^2 \atop x^3}\right] + \tilde\measuredangle^\kappa\left[x^1\,{x^3 \atop x^4}\right] + \tilde\measuredangle^\kappa\left[x^1\,{x^4 \atop x^2}\right] \le 2\cdot\pi,$$

or at least one of the three model angles $\tilde\measuredangle^\kappa\left[x^1\,{x^i \atop x^j}\right]$ is undefined [BGP 92].

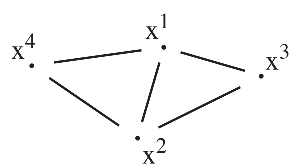


(b) A quadruple of points x^1, x^2, x^3, x^4 in a metric space $\mathcal{X}$ satisfies $(2+2)$ - *point κ-comparison*, in brief $\boxed{(2+2)^\kappa}$, if

(i) either $\tilde\measuredangle^\kappa\left[x^1\,{x^3 \atop x^4}\right] \le \tilde\measuredangle^\kappa\left[x^1\,{x^3 \atop x^2}\right] + \tilde\measuredangle^\kappa\left[x^1\,{x^2 \atop x^4}\right]$,

(ii) or $\tilde\measuredangle^\kappa\left[x^2\,{x^3 \atop x^4}\right] \le \tilde\measuredangle^\kappa\left[x^2\,{x^3 \atop x^1}\right] + \tilde\measuredangle^\kappa\left[x^2\,{x^1 \atop x^4}\right]$,

or at least one of the six model angles $\tilde\measuredangle^\kappa\left[x^i\,{x^j \atop x^k}\right]$ is undefined [AKP].



(c) $\mathcal{X} \in \mathrm{CAT}^\kappa \Leftrightarrow$ quadruples satisfy $\boxed{(2+2)^\kappa}$ [AKP].

(c) $\mathcal{X} \in \mathrm{CBB}^\kappa \Leftrightarrow$ quadruples satisfy $\boxed{(1+3)^\kappa}$ [BGP 92].

(d) These definitions $\Rightarrow$ CAT^κ, CBB^κ closed under GH–convergence to a complete metric space.

Exercise 2.4. Show that a complete π^κ-intrinsic space in which quadruples satisfy $\boxed{(2+2)^\kappa}$ is π^κ-geodesic.

3. Globalization I

Globalization theorems are power sources for Alexandrov geometry. We look at extensions of these theorems and ideas of proof in a later section.

(1) Let $\mathcal{X}$ be a locally intrinsic space.

(a) X has *curvature* $\geq \kappa$, written $\operatorname{curv} \mathcal{X} \geq \kappa$, if all quadruples lying in some neighborhood of p satisfy $(1+3)$-point κ-comparison;

(b) X has *curvature* $\leq \kappa$, written $\operatorname{curv} \mathcal{X} \leq \kappa$, if all quadruples lying in some neighborhood of p satisfy $(2+2)$-point κ-comparison.

(c) Example: Let X be a Riemannian manifold without boundary. Then $\operatorname{curv} X \geq \kappa \ (\leq \kappa) \ \Leftrightarrow \ \sec X \geq \kappa \ (\leq \ \kappa)$.

(2) (a) Lemma: $\mathcal{X} \in \mathrm{CBB}^\kappa$, $\dim \mathcal{X} = 1 \ \Leftrightarrow \ \mathcal{X}$ isometric to circle or closed, connected subset of $\mathbf{R}$.

(b) *CBB globalization theorem*[2]. Let $\mathcal{X}$ be a complete intrinsic space with $\operatorname{curv} \mathcal{X} \geq \kappa$. Then $\mathcal{X} \in \mathrm{CBB}^\kappa$. If $\dim \mathcal{X} > 1$, all triangles have perimeter $\leq 2 \cdot \pi^\kappa$.

(c) This theorem was proved by Alexandrov in dimension 2 [A 46], and by Toponogov for n-dimensional Riemannian manifolds [T 59]. In the above generality, the theorem first appears in [BGP 92]; simplifications and modifications were given in [Pl 96], as well as [S 93], [BB I01].

(d) Only now (with triangles of perimeter $> 2 \cdot \pi^\kappa$ ruled out) can we prove: If $\mathcal{X} \in \mathrm{CBB}^\kappa$ then $\mathcal{X} \in \mathrm{CBB}^{\kappa'}$ for all $\kappa' \leq \kappa$.

(3) (a) Upper curvature bounds yield more complicated globalization.

(b) Ex. 2.3 (a), applied repeatedly to a patchwork of small triangles, gives: *Alexandrov patchwork.* $\mathcal{X} \in \mathrm{CAT}^\kappa \ \Leftrightarrow \ \operatorname{curv} \mathcal{X} \leq \kappa$ and geodesics joining points at distance $< \pi^\kappa$ exist, are unique, and vary continuously with their endpoint pairs.

(c) *Hadamard-Cartan theorem.*[3] If $\mathcal{X}$ is a complete, simply connected intrinsic space satisfying $\operatorname{curv} \mathcal{X} \leq \kappa \leq 0$, then $\mathcal{X} \in \mathrm{CAT}^\kappa$.

(4) (a) Complete intrinsic spaces of $\operatorname{curv} \geq \kappa$ are closed under GH–convergence to a complete metric space, by CBB^κ globalization theorem and §2.(6)(d).

---

[2] This theorem does not assume finite dimension.

[3] The Hadamard–Cartan theorem was first stated for geodesic spaces $\mathcal{X}$ by Gromov [G 87, p.119]. A detailed proof when $\mathcal{X}$ is proper was given by Ballmann [Ba 85]. A proof in the non-proper case was given by Alexander-Bishop [AB 90]. Kleiner, Bridson-Haefliger observed that this proof extends to intrinsic spaces.

(b) Not so for curv $\leq \kappa$! e.g. one-sheeted hyperboloids with waist circumferences $\to 0$ satisfy curv ≤ 0. GH–limit $=$ double cone (which does not have unique geodesics in any neighborhood of the vertex).

4. CBB application: orbit space

Exercise 4.1. [BGP 92] Suppose curv $\tilde{\mathcal{X}} \geq \kappa$ and a group Γ acts on $\tilde{\mathcal{X}}$ by isometries with closed orbits. Show $\mathrm{curv}(\tilde{\mathcal{X}}/\Gamma) \geq \kappa$.

(1) (a) *Danzer-Grünbaum theorem* [DG 62]. (*Erdös' problem*[E 57].) If $x^1, \ldots, x^m \in \mathbf{E}^n$ have all angles $\leq \pi/2$, then $m \leq 2^n$. If $m = 2^n$, the x^i are vertices of a right parallelepiped.

(b) *Danzer-Grünbaum-Perelman theorem.* If $\mathcal{X} \in \mathrm{CBB}^0$, $\dim \mathcal{X} = n$, $m = \#$ extremal points, then $m \leq 2^n$.

(c) *Corollary.* For a discrete isometric action on $\mathbf{E}^n$, the number of isolated singular orbits is $\leq 2^n$.

(2) (a) Here p is an *extremal point* of $\mathcal{X} \in \mathrm{CBB}^\kappa$ if the "unit tangent sphere" $\Sigma_p\mathcal{X}$ has diameter $\leq \pi/2$ (see §8).

(b) Given a discrete isometric action by Γ on $\mathbf{E}^n$, $\mathcal{X} = \mathbf{E}^n/\Gamma \in \mathrm{CBB}^0$. An extremal point of $\mathcal{X}$ corresponds to an isolated singular orbit, i.e. the image of an isolated fixed point in $\mathbf{E}^n$ of some subgroup of Γ.

(c) *Proof of Corollary.* Immediate from (b) and D-G-P theorem.

(d) *Lebedeva's theorem* [Lb11]. If $\mathcal{X} \in \mathrm{CBB}^0$ has dimension n and 2^n extremal points, $\mathcal{X}$ is isometric to $\mathbf{E}^n/\Gamma$ for a discrete, cocompact isometric action by Γ.

(e) Examples of (d): Euclidean rectangular solid; tetrahedral surface glued from four congruent Euclidean triangles. [4]

[4] Source for this section: [Lb11].

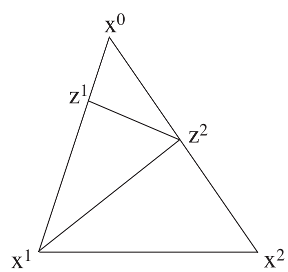
5. CAT construction: gluing

(1) (a) We say $\mathrm{CS} \subset \mathcal{X}$ is *π^κ-convex* if $|x\,y| < \pi^\kappa \ \Rightarrow\ \exists\,[x\,y]$, and all $[x\,y] \subset \mathrm{CS}$.

(b) CAT^κ is closed under gluing on isometric π^κ-convex subsets:

Reshetnyak gluing theorem [R 60]. Consider $\mathcal{X}^1, \mathcal{X}^2 \in \mathrm{CAT}^\kappa$, closed π^κ-convex subsets $\mathrm{CS}^i \subset \mathcal{X}^i$, and an isometry $\iota : \mathrm{CS}^1 \to \mathrm{CS}^2$ of CS^1 onto CS^2. If $\mathcal{X} = \mathcal{X}^1 \sqcup_\iota \mathcal{X}^2$ is the intrinsic space obtained by gluing $\mathcal{X}^i$ along ι, then $\mathcal{X} \in \mathrm{CAT}^\kappa$.

(c) *Proof assuming $\mathcal{X}$ is geodesic.* For triangle $[x^0x^1x^2]$ of perimeter $< 2\cdot\pi^\kappa$ in $\mathcal{X}$, we may suppose $x^0 \in \mathcal{X}^1$ and $x^1, x^2 \in \mathcal{X}^2$. Choose points z^1, z^2 in the gluing set, and lying on sides $[x^0x^1]$ and $[x^0x^2]$ respectively. Then $[x^0z^1z^2] \subset \mathcal{X}^1$, and $[x^1z^1z^2] \subset \mathcal{X}^2$, $[x^1z^2x^2] \subset \mathcal{X}^2$. Now apply Alexandrov "snap-out" Lemma twice to side-by-side model triangles.



Exercise 5.1. (Bishop [Bi08], Petrunin)

(i) Let $\mathcal{P}$ be a polygon in $\mathbf{E}^2$ carrying its intrinsic metric. Show $\mathcal{P} \in \mathrm{CAT}^0$.

(ii) Let $\mathcal{X}$ be the completion in the intrinsic metric of an open, simply connected, proper subset $\check{\mathcal{X}}$ of $\mathbf{E}^2$. Show $\mathcal{X} \in \mathrm{CAT}^0$.

(2) (a) Dually, CBB^κ is closed under gluing on isometric boundaries, e.g. closed balls in $\mathbf{E}^n$ and $\mathbf{H}^n$, of radius $\sinh r$ and r respectively, glued on their (isometric) boundary spheres.

Petrunin gluing theorem [**?**]. Consider $\mathcal{X}^1, \mathcal{X}^2 \in \mathrm{CBB}^\kappa$, and an isometry $\iota : \partial\mathcal{X}^1 \to \partial\mathcal{X}^2$ of $\partial\mathcal{X}^1$ onto $\partial\mathcal{X}^2$. If $\mathcal{X} = \mathcal{X}^1 \sqcup_\iota \mathcal{X}^2$ is the intrinsic space obtained by gluing $\mathcal{X}^i$ along ι, then $\mathcal{X} \in \mathrm{CBB}^\kappa$.

(b) If $\mathcal{X}^1 = \mathcal{X}^2$, this is *Perelman doubling theorem* [Pr91].

(c) CBB^κ gluing proof is harder than CAT^κ!

(d) Definition of $\partial\mathcal{X}$ for $\mathcal{X} \in \mathrm{CBB}^\kappa$ is given in §11.(5).

6. CAT application: billiards

Burago, Ferleger, and Kononenko [BFK 98] used CAT^0 geometry to solve a long-standing, celebrated open problem: *In a gas of m hard balls, $\exists$ a usniform bound on number of collisions.* Their method is by solving a more general billiards problem.[5]

(1) (a) Configuration space for m balls of radius $1/2$ in $\mathbf{E}^3$:

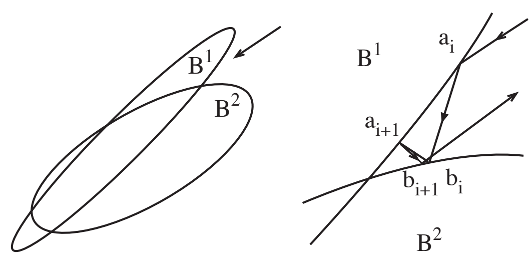
$$\bar{\mathcal{S}} = \mathbf{E}^{3m} - \{(x^1, \ldots, x^m) \in \mathbf{E}^{3m} : |x^i - x^j| < 1\}.$$

(b) Collisions $\sim \partial\bar{\mathcal{S}}$, motions of the balls $\sim$ billiards trajectories on table $\bar{\mathcal{S}}$.

(2) (a) More generally, for *any* convex bodies $B^i \subset \mathbf{E}^N$, consider billiarde in

$$\bar{\mathcal{S}} = \mathbf{E}^N - \cup_{i=1}^m B^i.$$

For simplicity, let $m = 2$, $\bar{\gamma}$ = billiard trajectory in $\bar{\mathcal{S}} = \mathbf{E}^N - (B^1 \cup B^2)$.



(b) Each time $\bar{\gamma}$ strikes ∂B^i, say at p, glue a new copy of $\mathbf{E}^N - B^i$ along ∂B^i. Let $\mathcal{S}$ be the resulting space. Then $\bar{\gamma}$ is the projection of a local geodesic γ in $\mathcal{S}$ that moves from one leaf to the next at each p.

(c) $\mathcal{S}$ is now our billiard table. Alas, $\mathcal{S}$ takes on curvature of ∂B^i! But by Reshetnyak gluing (§5), gluing in one copy of each B^i gives

$$\mathcal{S} \subset \mathcal{X} \in \mathrm{CAT}^0\,!$$

(d) γ remains a local geodesic in the new space $\mathcal{X}$, hence a geodesic by Ex. 2.2. Goal: upper bound on $\#$(strike points of γ). (Difficult case: γ almost parallel to intersection curve of B^1, B^2.)

(3) Core of proof (local argument): Assume γ lies in a neighborhood in $\mathcal{S}$ satisfying a necessary nondegeneracy condition on tangent planes to ∂B_i (see (b)).

(a) Let $a_1, \ldots, a_n \in B^1$ and $b_1, \ldots, b_n \in B^2$ be the strike points of γ (see figure). Set $d_i = |a_i\, b_i| + |b_i\, a_{i+1}| - |a_i\, a_{i+1}|$, and suppose d_j is the least of these.

(b) Let $z_i \in B^1 \cap B^2$ be the closest point of $B^1 \cap B^2$ to a_i. By nondegeneracy condition, $\exists\, C$ such that replacing $[a_i\, b_i] \cup [b_i\, a_{i+1}]$ by $[a_i\, z_i] \cup [z_i\, a_{i+1}]$ lengthens γ by $\leq C \cdot d_i$.

(c) Define a new path by replacing:

- $[a_j\, b_j] \cup [b_j\, a_{j+1}]$ by $[a_j\, z_j] \cup [z_j\, a_{j+1}]$;
- $[a_i\, b_i] \cup [b_i\, a_{i+1}]$ by $[a_i\,, a_{i+1}]$ for all $i \neq j$.

(d) Since new path intersects $B^1 \cap B^2$, length strictly increases. Hence
$C \cdot d_j - (n-1) \cdot d_j > 0$, so $n < C + 1$.

[5] Source for this section: [BB I01, §9.4]. The theorem applies to non-positively curved Riemannian manifolds, not just to $\mathbf{E}^n$.

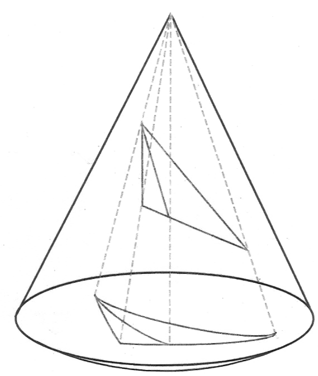
7. Cones

• Let $\mathcal{X}$ be a metric space.

Definition 7.1. Cone $\mathcal{X} = \big(\mathcal{X} \times [0,\infty)\big) \,/\, \big(\mathcal{X} \times \{0\}\big)$ with metric

$$|(x,s)(y,t)| \;=\; \sqrt{s^2 + t^2 - 2\cdot s\cdot t\cdot \cos\big(\min\{|x\,y|,\pi\}\big)}.$$

Exercise 7.2. Show $\mathcal{X} \in \mathrm{CAT}^1 \iff \text{Cone } \mathcal{X} \in \mathrm{CAT}^0$.



Similarly: If $\mathcal{X} \in \mathrm{CBB}^1$ and $\mathcal{X} \neq$ connected subset of $\mathbf{R}$ of length $> \pi^\kappa$ or circle of length $> 2\cdot\pi^\kappa$, then Cone $\mathcal{X} \in \mathrm{CBB}^0$. If Cone $\mathcal{X} \in \mathrm{CBB}^0$, then $\mathcal{X} = 2$ points or $\mathcal{X}$ is connected and $\mathcal{X} \in \mathrm{CBB}^1$.

8. First order structure

• Suppose $\mathcal{X} \in \mathrm{CAT}^\kappa$ or $\mathcal{X} \in \mathrm{CBB}^\kappa$.

(1) (a) Set $\Gamma_p\mathcal{X}$ = geodesics γ with $\gamma(0) = p$. Set $\gamma_1 \sim \gamma_2$ if this angle is 0. The *space of geodesic directions* is

$$\Sigma_p'\mathcal{X} = (\Gamma_p\mathcal{X}/\sim)$$

carrying the angle metric.

(b) The *direction space* $\Sigma_p\mathcal{X}$ is the metric completion of $\Sigma_p'\mathcal{X}$.

(c) The *tangent space* of $\mathcal{X}$ at p is $T_p\mathcal{X} = \mathrm{Cone}\,(\Sigma_p\mathcal{X})$.

(2) If *curv* $\mathcal{X} \le \kappa$, or *curv* $\mathcal{X} \ge \kappa$, $\mathcal{X}$ locally compact and all geodesics of $\mathcal{X}$ extendible, then

$$T_p\mathcal{X} \;=\; \text{GH–lim}_{\lambda\to\infty}\big(\lambda\mathcal{X}, p\big).$$

Exercise 8.1. Give examples of $\mathcal{X} \in \mathrm{CAT}^\kappa$ and $p \in \mathcal{X}$, where:

(i) the limit in (2) does not exist;

(ii) the limit in (2) exists but is not a cone over a metric space.

Theorem.

(i) [BGP 92] curv $\mathcal{X} \ge \kappa \;\Rightarrow\; T_p\mathcal{X} \in \mathrm{CBB}^0$.

(ii) [N 95] curv $\mathcal{X} \le \kappa \;\Rightarrow\; T_p\mathcal{X} \in \mathrm{CAT}^0$.

9. Riemannian examples

- Let $\mathcal{X}$ = n-dimensional Riemannian manifold ($n \geq 2$), possibly with $\partial\mathcal{X} \neq \emptyset$.

- For $p \in \partial\mathcal{X}$, denote the principal curvatures of $\partial\mathcal{X}$ at p by

$$k_1(p) \leq k_2(p) \leq \ldots \leq k_{n-1}(p).$$

Convention: $k_i(p) > 0$ for $\mathcal{X}$ = closed ball in $\mathbf{E}^n$;
$k_i(p) < 0$ for $\mathcal{X}$ = complement of open ball in $\mathbf{E}^n$.

(1) (a) $\operatorname{curv}\mathcal{X} \geq \kappa \quad\Leftrightarrow\quad \sec\mathcal{X} \geq \kappa$, and $\partial\mathcal{X}$ convex (all $k_i(p) \geq 0$).
(*sec* denotes sectional curvature.)

(b) *Buyalo convex hypersurface theorem* [B 76, AKP 08]. Suppose $\partial\mathcal{X} = \emptyset$, $\sec\mathcal{X} \geq \kappa$, and $\mathcal{C} \subset \mathcal{X}$ is a convex hypersurface with its intrinsic metric, i.e. $\mathcal{C}$ bounds a convex set. Then $\mathcal{C} \in \mathrm{CBB}^\kappa$.

(c) *Kosovskiĭ gluing theorem for* curv $\geq \kappa$ [Ko02-1]. Consider $\mathcal{X}^1, \mathcal{X}^2$ with $\sec\mathcal{X}^i \geq \kappa$, and an isometry $\iota : \partial\mathcal{X}^1 \to \partial\mathcal{X}^2$ of $\partial\mathcal{X}^1$ onto $\partial\mathcal{X}^2$. Set $\mathcal{X} = \mathcal{X}^1 \sqcup_\iota \mathcal{X}^2$. Then $\operatorname{curv}\mathcal{X} \geq \kappa \Leftrightarrow II(\partial\mathcal{X}^1) + II(\partial\mathcal{X}^2)$ is positive semi-definite.

(2) (a) *A.-Berg–Bishop characterization theorem for manifolds-with-boundary* [ABB 93]. $\operatorname{curv}\mathcal{X} \leq \kappa \quad\Leftrightarrow\quad \sec\mathcal{X} \leq \kappa$ and $\sec(\partial\mathcal{X})(P) \leq \kappa$ for any tangent 2-plane P on which $\partial\mathcal{X}$ is strictly concave, i.e. the second fundamental form $II(\partial\mathcal{X})$ is negative definite.

(b) *Corollary.* Suppose $\mathcal{X} \subset \overline{\mathcal{X}}$ where $\overline{\mathcal{X}}$ is an n-dimensional Riemannian manifold with $\partial\overline{\mathcal{X}} = \emptyset$ and $\sec\overline{\mathcal{X}} \leq \kappa$. If $k_2(p) \geq 0$ for all $p \in \partial\mathcal{X}$, then $\operatorname{curv}\mathcal{X} \leq \kappa$.

(c) *Kosovskiĭ gluing theorem for* curv $\leq \kappa$ [Ko02-2, Ko04]. Consider $\mathcal{X}^1, \mathcal{X}^2$ with $\sec\mathcal{X}^i \leq \kappa$, and an isometry $\iota : \partial\mathcal{X}^1 \to \partial\mathcal{X}^2$ of $\partial\mathcal{X}^1$ onto $\partial\mathcal{X}^2$. Set $\mathcal{X} = \mathcal{X}^1 \sqcup_\iota \mathcal{X}^2$. If $II(\partial\mathcal{X}^1) + II(\partial\mathcal{X}^2)$ is negative semi-definite, and $\sec(\partial\mathcal{X}^i)(P) \leq \kappa$ for any tangent 2-plane P on which $II(\partial\mathcal{X}^1)$ and $II(\partial\mathcal{X}^1)$ are both negative definite, then $\operatorname{curv}\mathcal{X} \leq \kappa$. The theorem extends to multiple $\mathcal{X}^i$ all of whose whose boundaries are isometric.

10. Convex functions

• Let $\mathcal{X}$ be a metric space.

(1) (a) For locally Lipschitz $f : \mathcal{X} \to \mathbf{R}$, write

$$f'' + \kappa f \geq c \quad (\leq c)$$

if $(*)$ $(f \circ \gamma)'' + \kappa \cdot (f \circ \gamma) \geq c \quad (\leq c)$ for every unitspeed geodesic γ.

(b) $(*)$ is in barrier sense, i.e. for each t, some y such that $y'' + \kappa \cdot y = c$ coincides with $(f \circ \gamma)$ at t, and satisfies $(f \circ \gamma) \geq y$ (respectively $\leq y$) on a sufficiently short open interval about t (*support property*).

(c) Then f is *semiconvex (semiconcave)*: $\exists$ constant generalized lower (upper) bound on f'' along geodesics. (f semiconvex $\Rightarrow (-f)$ semiconcave.)

(2) (a) Set

$$\mathrm{md}^\kappa(x) = \begin{cases} x^2/2, & \kappa = 0 \\ (1/\kappa)(1 - \cos\sqrt{\kappa}x), & \kappa > 0 \\ (1/\kappa)(1 - \cosh\sqrt{-\kappa}x), & \kappa < 0. \end{cases}$$

$$\mathrm{sn}^\kappa(x) = (\mathrm{md}^\kappa)'(x) = \begin{cases} x, & \kappa = 0 \\ (1/\sqrt{\kappa})\sin(\sqrt{\kappa}x), & \kappa > 0 \\ (1/\sqrt{-\kappa})\sinh(\sqrt{-\kappa}x), & \kappa < 0. \end{cases}$$

(b) In $\mathcal{M}(\kappa)$, $f = \mathrm{md}^\kappa \circ \mathrm{dist}_p$ satisfies $f'' + \kappa f = 1$.

(c) In $\mathcal{M}(\kappa)$, $f = \mathrm{sn}^\kappa \circ \mathrm{dist}_{[x\,y]}$ satisfies $f'' + \kappa f \geq 0$.

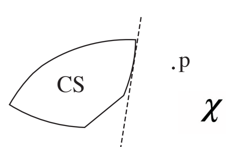
(3) Set $\mathrm{B}(A, r) = \{q \in \mathcal{X} : |q\,A| < r\}$ for $A \subset \mathcal{X}$. By (2)(b) and $\boxed{\text{Point-Side}}$,

(a) $\mathcal{X} \in \mathrm{CAT}^\kappa \;\Leftrightarrow f = \mathrm{md}^\kappa \circ \mathrm{dist}_p$ satisfies $f'' + \kappa f \geq 1$ on $\mathrm{B}(p, \pi^\kappa)$.

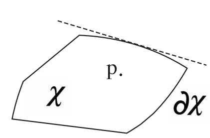
(b) $\mathcal{X} \in \mathrm{CBB}^\kappa \;\Leftrightarrow f = \mathrm{md}^\kappa \circ \mathrm{dist}_p$ satisfies $f'' + \kappa f \leq 1$ on $\mathrm{B}(p, \pi^\kappa)$.

(4) (a) On the other hand, (2)(c) relates to the following theorems:

(b) *Theorem.* Let $\mathcal{X} \in \mathrm{CAT}^\kappa$, and $\mathrm{CS} \subset \mathcal{X}$ be π^κ-convex. Then $f = \mathrm{sn}^\kappa \circ\ \mathrm{dist}_{\mathrm{CS}}$ satisfies $f'' + \kappa f \geq 0$ on $\mathrm{B}(\mathrm{CS}, \pi^\kappa/2)$.



(c) *Perelman concavity theorem.* If $\mathcal{X} \in \mathrm{CBB}^\kappa$, then $f = \mathrm{sn}^\kappa \circ\ \mathrm{dist}_{\partial X}$ satisfies $f'' + \kappa f \leq 0$. (See §11.(5) for definition of $\partial\mathcal{X}$.)



(5) (a) *Lemma.* Let $\operatorname{curv} \mathcal{X} \geq \kappa$ or $\operatorname{curv} \mathcal{X} \leq \kappa$, and $f : X \to \mathbf{R}$ be a locally Lipschitz function such that $(f \circ \gamma)^+(0)$ exists for every geodesic γ with $\gamma(0) = p$. Then $\exists!$ linearly homogeneous, Lipschitz map

$$\mathrm{d}_p f : \mathrm{T}_p\, X \to \mathbf{R}$$

such that $(\mathrm{d}_p f)(x) = (f \circ \gamma)^+(0)$ when γ is a geodesic with $\gamma^+(0) = x$.

(b) Let $\operatorname{curv} \mathcal{X} \geq \kappa$ or $\operatorname{curv} \mathcal{X} \leq \kappa$, and $f : X \to \mathbf{R}_{\geq 0}$ be a locally Lipschitz semiconcave function ((1)(c)). Then:

(i) $\mathrm{d}_p f$ exists and is concave.

(ii) The *gradient* $\nabla_p f \in T_p X$ exists, where $\nabla_p f = 0_p$ if $\mathrm{d}_p f \leq 0$, and otherwise

$$\nabla_p f = (\mathrm{d}_p f)(u_{\max}) \cdot u_{\max},$$

for unique $u_{\max} \in \Sigma_p \mathcal{X}$ where $(\mathrm{d}_p f) \,|\, \Sigma_p X$ takes its maximum.

(c) Semiconcave functions capture much of the geometry of Alexandrov spaces. Their *gradient curves* exist and are unique, and the corresponding gradient flow is locally Lipschitz. Gradient curves of semiconcave functions were introduced in [PP 94] (for $\operatorname{curv} \geq \kappa$), and their properties developed by Lytchak [Lt 05] (for both $\operatorname{curv} \geq \kappa$ and $\operatorname{curv} \leq \kappa$) and Petrunin [P 07].

Exercise 10.1. Let $\mathcal{X}$ be a closed triangular region in $\mathbf{E}^2$. Describe the gradient curves of $f = dist_{\partial X}$.

11. Local structure

(1) (a) Notions of metric space *dimension* include Hausdorff, topological (covering), and *geometric*: the largest number of times you can pass to direction space without getting the empty set.

(2) $\mathcal{X} \in \mathrm{CBB}^\kappa$ has nice local structure [BGP 92, Pr91, Pr93] :

(a) $\mathcal{X}$ is locally compact, hence proper and geodesic.

(b) $\dim_{\mathrm{Haus}} \mathcal{X}$ is an integer.

(c) All extant notions of $\dim \mathcal{X}$ agree.

(d) There is an open dense set of points p such that p has a neighborhood bi-Lipschitz homeomorphic to an open region in $\mathbf{R}^n$.

(e) If $\dim \mathcal{X} = n$, then $T_p\mathcal{X} \in \mathrm{CBB}^0$ has dimension n and $\Sigma_p\mathcal{X} \in \mathrm{CBB}^1$ has dimension $n-1$ for all $p \in \mathcal{X}$.

(f) The set of points p at which $\Sigma_p\mathcal{X}$ is isometric to $\mathbf{S}^{n-1}$ is dense, and a countable intersection of open dense sets.

(3) Kleiner [Kl99] introduced geometric dimension and studied its relation to other notions of $\dim \mathcal{X}$ for $\mathcal{X} \in \mathrm{CAT}^\kappa$ (a more complicated situation than CBB^κ). e.g. Conjecture: $\dim_{\mathrm{Geom}} \mathcal{X} = \dim_{\mathrm{Top}} \mathcal{X}$ (true for separable $\mathcal{X}$).

Exercise 11.1. Construct a metric tree $\mathcal{X}$ such that $\mathcal{X}^c$ has infinite Hausdorff dimension, where $\mathcal{X}^c$ is the metric completion of $\mathcal{X}$. (Note that $\mathcal{X}^c \in \mathrm{CAT}^0$ by Ex. 2.4. Also $\dim_{Top} \mathcal{X} = 1$.)

(4) (a) A key tool in (3) is the *barycentric simplex* Θ_f of $p^0, \ldots, p^k \in \mathcal{X}$ [Kl99]. Set $f=(f^0, \ldots, f^k)$, $f^i = \mathrm{md}^\kappa \circ \mathrm{dist}_{p^i}$ (§10.(3)(a)). Let $\Delta^k = \{\mathbf{x} \in \mathbf{R}^k : \sum_{i=0}^k x^i = 1,\, x^i \geq 0\}$, $\mathbf{x} = (x^0, \ldots, x^k)$. Define $\Theta_f : \Delta^k \to \mathcal{X}$ by

$$\Theta_f(\mathbf{x}) = \mathrm{MinPoint} \sum_{i=0}^k x^i \cdot f^i.$$

(b) The theorems of (2) depend on analogous constructions, with detailed study of strutting point arrays ("strainers") [BGP 92, Pr91, Pr93].

(5) Definition of $\partial \mathcal{X}$ for $\mathcal{X} \in \mathrm{CBB}^\kappa$ (by induction on dimension):

(a) If $\dim \mathcal{X} = 1$, then $\partial \mathcal{X} =$ topological boundary of $\mathcal{X}$ (so $\partial \mathcal{X} = 1$ or 2 points, or $\partial \mathcal{X} = \emptyset$).

(b) If $\dim \mathcal{X} = n > 1$, then

$$\partial \mathcal{X} \;=\; \{\, p \in \mathcal{X} \,:\, \partial\big(\Sigma_p \mathcal{X}\big) \neq \emptyset \,\}.$$

(1) (a) Let $\mathcal{B}$, $\mathcal{F}$ be intrinsic spaces, and $f\colon \mathcal{B} \to \mathbf{R}_{\geq 0}$ be continuous. Define $(\mathcal{B} \times \mathcal{F})/\sim$ by identifying elements of $\{p\} \times \mathcal{F}$ when $f(p) = 0$ (we identify this class with p). A curve $\gamma : J \to (\mathcal{B} \times \mathcal{F})/\sim$ determines $\gamma_{\mathcal{B}} : J \to \mathcal{B}$ and $\gamma_{\mathcal{F}} : J_+ \to \mathcal{F}$ for $J_+ = J - J_0$, $J_0 = (f \circ \gamma_{\mathcal{B}})^{-1}(0)$. Suppose $\gamma_{\mathcal{B}}$ and $\gamma_{\mathcal{F}}$ are Lipschitz.

(b) Writing $J_+ = \sqcup J_i$, set

$$\operatorname{length}\gamma \;=\; \sum_i \int_{J_i} \sqrt{v_{\mathcal{B}}^2 + (f \circ \gamma_{\mathcal{B}})^2 \cdot v_{\mathcal{F}}^2} \;\;+\;\; \operatorname{length}(\gamma_{\mathcal{B}}|J_0),$$

where $\int$ is Lebesgue integral, and $v_{\mathcal{B}}$, $v_{\mathcal{F}}$ are speeds of $\gamma_{\mathcal{B}}$, $\gamma_{\mathcal{F}}$ (defined almost everywhere on their domains).

(c) The *warped product* $\mathcal{X} = \mathcal{B} \times_f \mathcal{F}$ is the corresponding intrinsic space.

(2) *Examples:* (a) $\mathcal{B} \times_1 \mathcal{F} = \mathcal{B} \times \mathcal{F}$.

(b) $\mathcal{X} = \{r \in [0,\, c < \pi]\} \times_{\sin r} \mathbf{S}^n(1)$ is a metric ball in $\mathbf{S}^{n+1}(1)$.

$c \leq \pi/2 \;\Rightarrow\; \mathcal{X} \in \mathrm{CBB}^1\,;\;\; \mathcal{X} \in \mathrm{CAT}^1$.
$c > \pi/2 \;\Rightarrow\; \mathcal{X} \notin \mathrm{CBB}^\kappa$ for any $\kappa\,;\;\; \mathcal{X} \in \mathrm{CAT}^{1/\sin^2 c}$.

(c) *κ-cone of $\mathcal{F}$*: $\operatorname{Cone}_\kappa \mathcal{F} = \{r \in \mathbf{I}^\kappa\} \times_{\mathrm{sn}^\kappa r} \mathcal{F}$, $\mathbf{I}^\kappa = [0,\infty)$ if $\kappa \leq 0$, or $[0, \pi^\kappa]$.

Spherical suspension of $\mathcal{F}$: $\operatorname{Cone}_1 \mathcal{F}$.

κ-join of $\mathcal{F}^1$, $\mathcal{F}^2$: $\operatorname{Join}_\kappa\big(\mathcal{F}^1, \mathcal{F}^2\big) = \big(\{r \in \mathbf{I}^\kappa\} \times_{\mathrm{sn}^\kappa r} \mathcal{F}^1\big) \times_{\mathrm{cs}^\kappa \circ \mathrm{proj}_r} \mathcal{F}^2$,
where $\mathrm{cs}^\kappa = (\mathrm{sn}^\kappa)'$, and proj_r is projection to first factor.

(d) Let $\mathcal{M}^0(\kappa)$ be a point if $\kappa \leq 0$; two points at distance π^κ if $\kappa > 0$. If $n \geq 1$,

$$\mathcal{M}^n(\kappa) = \operatorname{Cone}_\kappa \mathcal{M}^{n-1}(1), \qquad \mathcal{M}^{m+n+1}(\kappa) = \operatorname{Join}_\kappa\big(\mathcal{M}^m(1), \mathcal{M}^n(\operatorname{sgn}\kappa)\big).$$

Exercise 12.1. (i) Let $\mathcal{B}$ be a closed triangular region in $\mathbf{E}^2$. Describe $\mathcal{X} = \mathcal{B} \times_f \mathbf{S}^1(1)$, where $f = \operatorname{dist}_{\partial\mathcal{B}}$.

(ii) Let $\mathcal{X} = \mathbf{E}^2 \times_f \mathbf{S}^1(1)$, where $f = \operatorname{dist}_0$. Find $\Sigma_{\bar{0}}\, \mathcal{X}$.

Exercise 12.2. Show the spherical join, $\operatorname{Join}_1\big(\mathcal{X}^1, \mathcal{X}^2\big)$, satisfies

$$\operatorname{Cone}\, \operatorname{Join}_1\big(\mathcal{X}^1, \mathcal{X}^2\big) = \operatorname{Cone} \mathcal{X}^1 \times \operatorname{Cone} \mathcal{X}^2.$$

(3) (a) $\mathcal{B} \times \{\varphi_0\}$ is isometric to $\mathcal{B}$, and isometrically embedded in $\mathcal{X}$.

(b) If $f(p_0) \neq 0$, then $\{p_0\} \times \mathcal{F}$, with its intrinsic metric, is homothetic to $\mathcal{F}$ with multiplier $f(p_0)$.

(c) By length formula, geodesics of $\mathcal{X}$ seek lower f-values. e.g. If $f(p_0) > 0$ is local min of f, then $\{p_0\} \times F$ is isometrically embedded in $\mathcal{X}$. In this case, $\operatorname{curv} \mathcal{X} \leq \kappa \;\Rightarrow\; \operatorname{curv} \mathcal{F} \leq \kappa_{\mathcal{F}} = \kappa \cdot f(p_o)^2$, and same for $\geq$.

13. Warped product examples

• Suppose $\mathcal{X} = \mathcal{B} \times_f \mathcal{F}$ for $\mathcal{B}$, $\mathcal{F}$ intrinsic; $f : \mathcal{B} \to \mathbf{R}_{\geq 0}$ Lipschitz; $Z = f^{-1}(0) \neq \mathcal{B}$. Write $f \in \widehat{\mathcal{C}}^\kappa$ if $f'' + \kappa f \leq 0$, and $f \in \check{\mathcal{C}}^\kappa$ if $f'' + \kappa f \geq 0$.

(1) *A.-Bishop characterization theorem for warped products* [AB 98, AB 04, AB 14]:

$\boxed{\mathcal{X} \in \mathrm{CBB}^\kappa \iff :}$ [6]

(a) $\mathcal{B} \in \mathrm{CBB}^\kappa$ and $f \in \widehat{\mathcal{C}}^\kappa$.

(b) $\mathcal{F} \in \mathrm{CBB}^{\kappa_\mathcal{F}}$ where:

(i) if $Z = \emptyset$, $\kappa_F = \kappa \cdot (\inf f)^2$

(*Compare §12.(3)(c), or its asymptotic version.*)

(ii) if $Z \neq \emptyset$, $\kappa_F = \sup\{(f \circ \alpha)^+(0)^2\}$ for dist_Z-realizers α with footpoint $\alpha(0) \in Z$, $|\alpha^+(0)| = 1$.
e.g. $\mathcal{X} = \mathrm{Cone}_\kappa\, \mathbf{S}^n(1/c^2))$, $c \leq 1$.

$\boxed{\mathcal{X} \in \mathrm{CAT}^\kappa \iff :}$

(a) $\mathcal{B} \in \mathrm{CAT}^\kappa$ and $f \in \check{\mathcal{C}}^\kappa$.

(b) $\mathcal{F} \in \mathrm{CAT}^{\kappa_\mathcal{F}}$ where:

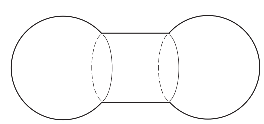
(i) if $Z = \emptyset$, $\kappa_F = \kappa \cdot (\inf f)^2$

(ii) if $Z \neq \emptyset$, $\kappa_F = \inf\{(f \circ \alpha)^+(0)^2\}$ for dist_Z-realizers α with footpoint $\alpha(0) \in Z$, $|\alpha^+(0)| = 1$.
e.g. $\mathcal{X} = \mathrm{Cone}_\kappa\, \mathbf{S}^n(1/c^2))$, $c \geq 1$.

OR (if less)

(iii) $\kappa_\mathcal{F} = \inf\{\kappa \cdot f(p)^2 : \mathrm{dist}_Z(p) \geq \varpi^\kappa/2\}$

(*e.g.* $\mathcal{X} =$ *dumbbell* $= I \times_f \mathbf{S}^n(1)$, $n \geq 2$, $I = [0, c > \pi]$; $f(x) = \sin x$ *for* $x \leq \pi/2 + \epsilon < c/2$; $f(x) = \sin\big(x - (c - \pi)\big)$ *for* $x \geq c - \pi/2 - \epsilon$; *otherwise* $f(x) = \sin(\pi/2 + \epsilon)$. *For* $\kappa = 1$, $\mathcal{X}$ *satisfies (1), (2)(ii), not (2)(iii).* $\mathcal{X} \notin \mathrm{CAT}^1$ *since* $\{c/2\} \times \mathbf{S}^n(1)$ *is isometrically embedded in* $\mathcal{X}$ *with curvature* $1/f(c/2)^2 > 1$.)



(c) *Gluing condition:*
$\mathcal{B}^\dagger \in \mathrm{CBB}^\kappa$, $f^\dagger \in \widehat{\mathcal{C}}^\kappa$ where $\mathcal{B}^\dagger =$
2 copies of $\mathcal{B}$ glued on $\mathrm{cl}\,(\partial\mathcal{B} - Z)$,
$f^\dagger =$ canonical extension of f to $\mathcal{B}^\dagger$.

(*e.g.* $\widetilde{\mathcal{B}} =$ *closed unit ball in* $\mathbf{E}^n$, $\tilde{f} = \mathrm{dist}_{\partial\mathcal{B}}$, $\mathcal{B} =$ *convex body in* $\tilde{\mathcal{B}}$, $f = \tilde{f}|B$. *Then* $\mathcal{B}, f$ *satisfy gluing condition for* $\kappa = 0$ $\Leftrightarrow$ $\mathcal{B} \cap$ *(any ray from* 0*) either has endpoint on* $\partial\tilde{\mathcal{B}}$, *or* $= \emptyset$.)

[6] For this theorem, exclude connected subsets of $\mathbf{R}$ of length $> \pi^\kappa$ and circles of length $> 2 \cdot \pi^\kappa$ from CBB^κ.

(2) *Application of warped product characterization theorem:*

(a) Gromov Q [G 87]: If $\mathcal{X} \in \mathrm{CAT}^0$ is an n-manifold, is $\mathcal{X}$ homeomorphic to $\mathbf{E}^n$?

(b) Davis-Januszkiewicz gave counterexamples by (complicated) hyperbolization process [DJ 92]. Ancel-Guilbault used hyperbolic trigonometry and structure theory for compact contractible manifolds to construct counterexamples [AG 97]. In fact these are hyperbolic joins (Join_{-1}).

(c) Q [AG 97]: If $\mathcal{B} \in \mathrm{CAT}^{-1}$, does $\mathcal{B} \times \mathbf{R}$ always carry a negatively curved metric whose levels are convex subsets?

(d) *Yes:* $\mathcal{B} \times_f \mathbf{R} \in \mathrm{CAT}^{-1}$ for any $f \in \check{\mathcal{C}}^{-1}$ (e.g. $f = \cosh d_p$ for $p \in \mathcal{B}$, or $f = \exp$ (Busemann function). These are the only warped product examples.

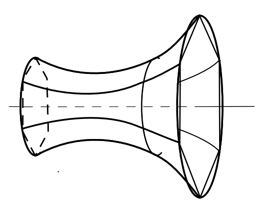
(3) *Proving "if" in* CAT^0 *characterization theorem:*

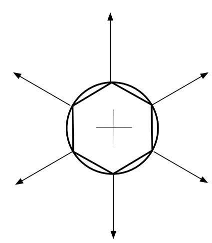
(a) Reduce to $\mathcal{F} = \mathcal{M}^2(\kappa)$. By §12.(2)(d), reduce to $\dim \mathcal{F} = 1$.

(b) (Take $\kappa = 0$.) Decompose $B \times \mathbf{R}$ into *strips* carrying the metric of

$$\{(p, u) : -\epsilon f(p) \leq u \leq \epsilon f(p)\},$$

gluing top boundary of one strip to bottom boundary of next to get approximating space $\mathcal{S}$. If $\mathcal{B} \in \mathrm{CBB}^0$ and f concave, then $\mathcal{S} \in \mathrm{CBB}^0$. But when $\mathcal{B} \in \mathrm{CAT}^0$ and f convex, $\mathcal{S}$ takes on positive curvature of gluing seams! In this case, at each gluing seam attach a "fin", i.e. a copy of $\{(p, u) : u \geq \epsilon f(p)\}$, to recover CAT^0 (compare billiards, §6.(2)(c)). See figure, where $\mathcal{X}$ is the simply connected cover of the illustrated surface.





(4) *Proving "only if" in characterization theorem:*

(a) CAT^κ has a rich theory of *subspaces*. (A problematical topic in CBB^κ.)[7] Say $\mathcal{U} \subset \mathcal{X}$ has *extrinsic curvature* $\leq A$ if intrinsic distance s in $\mathcal{U}$ and extrinsic distance r in $\mathcal{X}$ satisfy $s - r \leq \frac{A^2}{24} r^3 + o(r^3)$ for s sufficiently small. For subspaces $\mathcal{U} = \{p\} \times \mathcal{F}$, apply:
A.-Bishop Gauss-equation theorem [AB 06].
$$(\text{Extrinsic curvature } \mathcal{U}) \leq A \implies \operatorname{curv} \mathcal{U} \leq \kappa + A^2.$$

(b) CBB^κ allows a standard method of proof by induction on $n = \dim \mathcal{X}$: Prove for $n = 1$, then show the theorem passes from direction spaces $\Sigma_p \mathcal{X}$ (dimension $n-1$) to $\mathcal{X}$.

(c) Proof of CBB^κ gluing condition requires Petrunin incomplete-globalization theorem (§14.(2)(a)): Set $\mathcal{B}_0^\dagger = \mathcal{B}^\dagger - (f^\dagger)^{-1}(0)$. Then $(\mathcal{B}_0^\dagger)^c \cong \mathcal{B}^\dagger$. Thus it suffices to prove $\mathcal{B}_0^\dagger$ is geodesic, and $\operatorname{curv} \mathcal{B}_0^\dagger \geq \kappa$.

[7] Celebrated problem: does ∂S inherit a curvature bound if $S =$ convex set in $\mathcal{X} \in \mathrm{CBB}^\kappa$?

(1) • Let $\mathcal{X}$ = complete intrinsic space with $\operatorname{curv} \mathcal{X} \le \kappa$.

(a) *A.-Bishop no-conjugate-point theorem* [AB 90]. Let γ be a simple geodesic, $\ell(\gamma) < \pi^\kappa$. Then there is a space $\mathcal{N} \in \mathrm{CAT}^\kappa$, a geodesic $\hat\gamma$ in an open subset $\hat\Omega \subset \mathcal{N}$, and an open isometric embedding $\Phi : \hat\Omega \to \mathcal{X}$ with $\Phi \circ \hat\gamma = \gamma$. (If γ not simple, use isometric immersion.)

(b) Proof by Reshetnyak gluing (§ 5) of small CAT^κ balls along $\hat\alpha$.[8]

(c) Then local geodesics move homotopically as endpoints follow preassigned paths, stopping only if length reaches π^κ. Using Alexandrov patchwork (§3), can now prove Hadamard-Cartan theorem (§3) as well as the following (each leading to the next):

Radial lemma [AB 96]. If geodesics from p to points of $B(p, \pi^\kappa/2)$ exist, are unique and vary continuously, then $B(p, \pi^\kappa/2) \in \mathrm{CAT}^\kappa$.

Lifting globalization theorem [AKP]. $\exists\ \mathcal{B} \in \mathrm{CAT}^\kappa$ with $\mathcal{B} = B(\hat p, \pi^\kappa/2)$ for some $\hat p \in \mathcal{B}$, and a local isometry $\Phi : \mathcal{B} \to \mathcal{X}$ with $\Phi(\hat p) = p$, such that for any curve $\alpha : [0,1] \to \mathcal{X}$ with $\alpha(0) = p$ and length $\alpha < \pi^\kappa/2$, there is a unique path $\hat\alpha : [0,1] \to \mathcal{B}$ such that $\hat\alpha(0) = \hat p$ and $\Phi \circ \hat\alpha = \alpha$.

Generalized Hadamard–Cartan theorem [AKP] (due to Bowditch if $\mathcal{X}$ is proper [Bo95]). $\mathcal{X} \in \mathrm{CAT}^\kappa \iff \mathcal{X}$ is $2 \cdot \pi^\kappa$-simply connected, i.e. any closed curve of length $< 2 \cdot \pi^\kappa$ is null-homotopic through closed curves of length $< 2 \cdot \pi^\kappa$.

(2) (a) *Petrunin incomplete-globalization theorem* [P]. Let X be a geodesic space and X^c be its completion. If $\operatorname{curv} X \ge \kappa$, then $X^c \in \mathrm{CBB}^\kappa$.

[8] The no-conjugate-point theorem of [AB 90] applies to a larger class of spaces and required a different proof. This CAT^κ formulation is due to Lytchak.

15. Polyhedral examples

• $\mathcal{X}$ = spherical, Euclidean or hyperbolic polyhedral space, i.e. $\exists$ finite triangulation by simplices σ, σ isometric to simplex σ^m in $\mathcal{M}^m(\kappa)$, fixed κ, some m.[9]

(1) (a) A neighborhood of $p \in \mathcal{X}$ is isometric to a tip of $\mathrm{Cone}_\kappa \Sigma_p$.

(b) $\mathrm{Link}\,\sigma = \{\sigma' : \sigma \cap \sigma' = \emptyset;\ \sigma$ and σ' are faces of the same simplex$\}$.

(c) $\mathrm{Link}\,\sigma$ carries natural structure of spherical polyhedron.

(d) If p = interior point of k-simplex σ, then $T_p\mathcal{X} \cong \mathbf{E}^k \times (\mathrm{Cone}\ \mathrm{Link}\,\sigma)$, so $\Sigma_p\mathcal{X} \cong k$-th spherical suspension, $\big(\mathrm{Cone}_1\big)^k\,\mathrm{Link}\,\sigma$.

(2) (a) *Theorem.* $\mathcal{X} \in \mathrm{CBB}^\kappa \iff$ following conditions hold:

(i) $\mathcal{X}$ is *pure*, i.e. $\exists\ m \geq 0$ such that any σ = face of some σ^m.

(ii) $\mathrm{Link}\,\sigma^{m-1}$ = one or two points.

(iii) $\mathrm{Link}\,\sigma^k$, $k \leq m-2$, is connected.

(iv) $\mathrm{Link}\,\sigma^{m-2}$ isometric to circle of length $\leq 2\cdot\pi$ or closed interval of length $\leq \pi$.

(b) *Theorem.* $\mathrm{curv}\,\mathcal{X} \leq \kappa$

$\Leftrightarrow$ any connected component of any $\mathrm{Link}\,\sigma$ is $(2\cdot\pi)$-simply connected

$\Leftrightarrow$ any closed local geodesic in Link_σ has length at least $2\cdot\pi$.

[9] Source for this section: [AKP].

References

[AB 90] S. Alexander; R. Bishop, *The HadamardCartan theorem in locally convex spaces*, Enseignement Math., **36** (1990), 309320.

[AB 96] S. Alexander, R. Bishop, *Comparison Theorems for Curves of Bounded Geodesic Curvature in Metric Spaces of Curvature Bounded Above*, Differential Geometry and its Applications, **6** (1996), 67-86.

[AB 98] S. Alexander, R. Bishop, *Warped products of Hadamard spaces*, Manuscripta Math. **96** (1998), 487-505.

[AB 04] S. Alexander, R. Bishop, *Curvature bounds for warped products of metric spaces*. Geom. Funct. Anal. (GAFA), **14**(2004) 1143-1181, http://www.math.uiuc.edu/ sba/

[AB 06] S. Alexander, R. Bishop, *Gauss equation and injectivity radii for subspaces in spaces of curvature bounded above*, Geom.Dedicata **117**, (2006).

[AB 14] S. Alexander, R. Bishop, *Warped products admitting a curvature bound*, submitted.

[ABB 93] S. Alexander, I. Berg, R. Bishop, *Geometric curvature bounds in Riemannian manifolds with boundary*, Trans. A.M.S. **339** (1993), 703-716.

[AKP] S. Alexander, V. Kapovitch, A. Petrunin, *Alexandrov geometry*, in preparation.

[AKP 08] S. Alexander, V. Kapovitch, A. Petrunin, *An optimal lower curvature bound for convex hypersurfaces in Riemannian manifolds*, Illinois J. Math. **52** (2008), 1031-1033.

[AKP 11] S. Alexander, V. Kapovitch, A. Petrunin, *Alexandrov meets Kirszbraun*, Proc. Gokova Geometry-Topology Conference 2010, S. Akbulut, D. Auroux, T. Onder, eds., International Press (2011), 88- 109.

[A 46] A.D. Alexandrov, *Intrinsic Geometry of Convex Surfaces*, Gostekhizdat, Moscow 1946 (Russian). Translation in *Selected Works, Part II*, Chapman and Hall/CRC, Boca Raton, 2006.

[AG 97] *Interiors of compact contractible N-manifolds are hyperbolic ($N \geq 5$)*, J. Differential Geometry **45** (1997). 1-33.

[Ba 85] W. Ballman, *Singular spaces of non-positive curvature*, Sur les groupes Hyperboliques d'après Gromov, (E. Ghys and P. de la Harpe, eds.), Birkhuser, Basel , (1985), 189-201.

[Ba95] Ballmann, Werner Lectures on spaces of nonpositive curvature. With an appendix by Misha Brin. DMV Seminar, 25. Birkhuser Verlag, Basel, 1995. viii+112 pp. ISBN: 3-7643-5242-6 (Reviewer: Boris Hasselblatt) 53C21 (58F17)

[Bi08] R. Bishop, *The intrinsic geometry of a Jordan domain*, Int. Electron. J. Geom. **1** (2008), 3339.

[Bo95] B. Bowditch, *Notes on locally CAT(1) spaces in Geometric group theory*, (ed. R. Charney, M. Davis, M. Shapiro), de Gruyter (1995), 148.

[BH] M. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature*. Springer, Berlin (1999).

[BB I01] 9. Burago, D., Burago, Yu., Ivanov, S.: A Course in Metric Geometry, Graduate Studies in Mathematics, Vol. 33. American Mathematical Society, Providence, Rhode Island (2001).

[BFK 98] D. Burago, S. Ferleger, A. Kononenko, *Uniform estimates on the number of collisions in semi-dispersing billiards*. Ann. of Math. **147** (1998), 695708.

[BGP 92] Yu. Burago, M. Gromov, G. Perelman, *A.D. Aleksandrov spaces with curvatures bounded below* (Russian) Uspekhi Mat. Nauk **47** (1992), 351, 222; translation in Russian Math. Surveys **47** (1992), 1-58.

[B 76] S. Buyalo, *Shortest paths on convex hypersurfaces of a Riemannian manifold* (Russian) Zap. Naukn. Sem. LOMI **66** (1976), 114-132; translation in J. Soviet Math. **12** (1979), 73-85.

[BS 07] S. Buyalo, V. Schroeder, *Spaces of curvature bounded above*, Surveys in differential geometry **11**, Int. Press (2007), 295327.

[DG 62] L. Danzer, B. Grünbaum, Math. Z. (1962), 95-99.

[DJ 92] M. Davis. T . Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geom. **34** (1991), 347-388.

[E 57] P. Erdös, Michigan Math. J. (1957), 291-300.

[G 87] M. Gromov, *Hyperbolic groups*, Essays in Group Theory (S.M. Ger- sten, ed.), Math. Sciences Research Institute Publications **8**, Springer- Verlag, Berlin, 1987, 75264.

[Kl99] B. Kleiner, *The local structure of length spaces of curvature bounded above*, Math. Z. (1999), 409-456.

[Ko02-1] N. Kosovskiĭ *Gluing of Riemannian manifolds of curvature at least κ*, Alebra i Analiz **14** (2002), 140-157(Russian), St. Petersburg Math. J. **14** (2003), 467-478.

[Ko02-2] N. Kosovskiĭ *Gluing of Riemannian manifolds of curvature at most κ*, , St. Petersburg Math. J. **14** (2003), 765-773.

[Ko04] N. KosovskiĭGluing with branching of Riemannian manifolds of curvature at most κ, Alebra i Analiz **16** (2004), 132-145 (Russian).

[LS 97] U. Lang, V. Schroeder *Kirszbraun's theorem and metric spaces of bounded curvature*, G.A.F.A. **7** (1997), 535-560.

[Lb11] N. Lebedeva, *Alexandrov spaces with maximal number extremal points*, arXiv 1111.7253v2.

[Lt 05] A. Lytchak, *Open Map Theorem for Metric Spaces*, Algebra i Analiz **17** (2005), 139-159 (Russian); translation in St. Petersburg Math. J. **17** (2006), 477-491.

[N 95] I. Nikolaev, *The tangent cone of an Alexandrov Space of curvature $\leq K$*, Manuscripta Math. **86** (1995), 683-689.

[Pr91] G. Perelman, *Alexandrov's spaces with curvature bounded from below II*, preprint LOMI (1991).

[Pr93] G. Perelman, Elements of Morse theory on Aleksandrov spaces. (Russian) Algebra i Analiz 5 (1993), 232-241; translation in St. Petersburg Math. J, **5** (1994), 205-213.

[PP 94] G. Perelman, A. Petrunin, *Quasigeodesics and Gradient Curves in Alexandrov Spaces*, preprint (1994), www.math.psu.edu/petrunin.

[P 07] A. Petrunin, *Semiconcave functions in Alexandrov's geometry*, Surveys in differential geometry **11**, Int. Press (2007),

[P] A. Petrunin, *A globalization for non-complete but geodesic spaces*, arXiv:12083155v3.

[Pl 96] C. Plaut, *Spaces of Wald-Berestovskii curvature bounded below*, J. Geom. Anal. **6** (1996), 113-134.

[Pl 02] C. Plaut, *Metric spaces of curvature $\geq k$*. R. Daverman, R. Sher (eds.), *Handbook of Geometric Topology*, Elsevier, (2002), 819-898.

[S 93] Shiohama, K., An introduction to the geometry of Alexandrov spaces. Lecture Notes Series, 8. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993. ii+78 pp.

[R 60] Yu. Reshetnyak, *On the theory of spaces of curvature not greater than K*, Mat. Sb. **52** (1960), 789-798 (Russian). English translation: *On the theory of spaces of curvature not greater than K*, Siberian Math. J.

[R 68] Yu. Reshetnyak, *Nonexpanding maps in a space of curvature no greater than K*, Sibirskii Mat. Zh. 9 (1968), 918-928 (Russian). English translation: *Inextensible mappings in a space of curvature no greater than K*, Siberian Math. J. **9** (1968), 683-689.

[S 86] D. Scolozzi, *Un risultato di locale unicitá per Ie geodetiche su varietá con bordo,* Bollettino U.M.I **8** (1986), 309-327.

[T 59] V. Toponogov, *Riemannian spaces of curvature bounded below*, Usp. Mat. Nauk **14** (1959), 87130.