Geodesics on Lorentz Surfaces of Revolution in Minkowski Space

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IGL Project

Comparison Geometry

- Triangles are formed by shortest paths called geodesics.
- Geodesics are solutions to a differential equation in smooth spaces.
- In spaces of constant curvature (e.g. the sphere, Euclidean space, and Hyperbolic space) geodesics, triangles, and angles have well-known behavior.
- Comparison geometry is about comparing the geometry of spaces with curvature bounded above (CAT(κ)) or below (CBB(κ)) by κ to spaces with constant curvature κ.
- CBB = "fatter" triangles, CAT = "skinnier" triangles.

Figure: Two-dimensional spaces of constant positive, negative, and zero curvature, respectively.

Lorentz Geometry

- Lorentz manifolds model spacetime in relativity theory.
- They have one time dimension and one or more spatial dimensions.
- Mathematically defined with metric that is negative definite in one dimension and positive definite in the spatial dimensions.
- No notion of distance, but geodesics defined using metric.
- Three types of geodesics: timelike, spacelike, lightlike (null). This property is called causal character.

Figure: Tangent Space in Lorentz Geometry

Mathematics

- Start with warping function r(s) > 0. Produce unit speed profile curve (r(s), f(s)) by
  \[ f(s) = \int_0^s \sqrt{r(t)^2 + 1} \, dt. \]
- Revolve profile curve around z-axis, obtaining
  \[ F(\theta, s) = (r(s) \cos \theta, r(s) \sin \theta, f(s)). \]
- Metric in these coordinates is given by
  \[ \begin{pmatrix} r(s)^2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}. \]
- Geodesic equations are given by
  \[ \ddot{s} + 2r(s) \dot{s} \dot{\theta} = 0, \quad \dot{s} + r(s) \ddot{s} = 0. \]
- Timelike, spacelike, and lightlike are determined by \(|\dot{s}|^2 < 0, |\dot{s}|^2 > 0, |\dot{s}|^2 = 0\), respectively.
- Sectional curvature formula given by
  \[ K(\theta, s) = -\frac{r''(s)}{r(s)} \]

Facts

- Several statements are true of geodesics and can be seen with our work in MATLAB.
- All geodesics γ have constant "square speed" in the Lorentzian sense:
  \[ r(\gamma)^2 - \dot{s}^2 = ||\dot{\gamma}||^2 = \text{const.} \]
- Clairaut’s relation holds:
  \[ r(\gamma)^2 \dot{\theta} = \text{const.} \]
- One can prove that all (with the exception of equatorial) geodesics on CBB(κ) with κ > 0 fall into big bangs and big crunches in finite time. This is seen in the top right figure.
- All nonspace like geodesics on CAT(κ) spaces with κ < 0 have bounded θ, i.e. they do not spiral around indefinitely.

Future Work

- Lorentz surfaces of revolution are warped products of the form \((-S^1) \times_S S^1\).
- In the future, we would like to make animations in these spacetimes.
- Also, replace \(S^1\) with other Riemannian manifolds for more general warped products of the form \((-\mathbb{R}) \times_M M\).
- Study geodesics in warped products with \(M = S^2, \mathbb{H}^2\) or other interesting manifolds.

Comparison Curvature

- Left: geodesics of each causal character on surface with constant curvature −1.
- Right: geodesics of each causal character on surface with constant curvature +1.

Bounded Curvature

- Geodesics are solutions to a differential equation in smooth spaces.
- We use MATLAB for fast numerical computation.
- Focus on surfaces of revolution in Minkowski space.
- Z-axis represents time, angle is spatial position.

Future Project

- Make computer visualizations to aid research in Lorentz spaces with curvature bounds.
- Geodesic equations are difficult to solve numerically.
- Study geodesics in warped products with \(M = S^2, \mathbb{H}^2\) or other interesting manifolds.

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