Geodesics and Curvature in Lorentz Spaces

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Comparison Geometry

- Triangles formed by shortest paths called geodesics.
- In spaces of constant curvature (e.g. the sphere, Euclidean space, and hyperbolic space) geodesics, triangles, and angles have well-known behavior.
- Comparison geometry compares the geometry of spaces with curvature bounded above (CBA(κ)) or below (CBB(κ)) by κ to model spaces with constant curvature κ.
- CBB = "fatter" triangles, CBA = "skinnier" triangles.

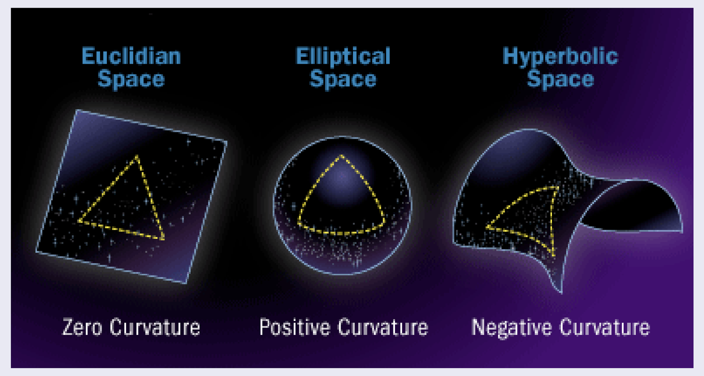
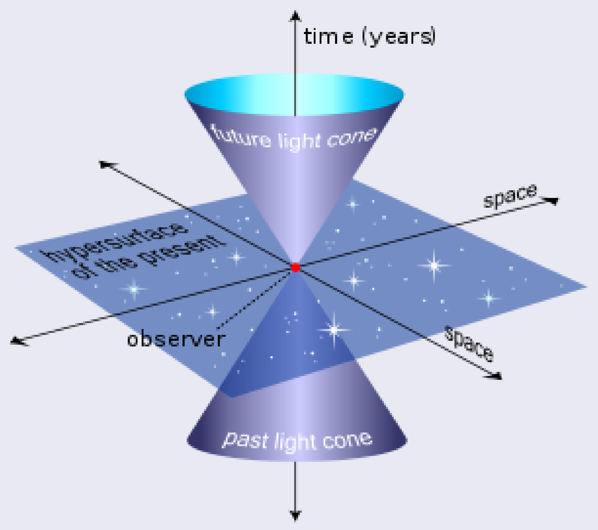


Figure: Two-dimensional spaces of constant zero, positive, and negative curvature, respectively.

Lorentz Geometry



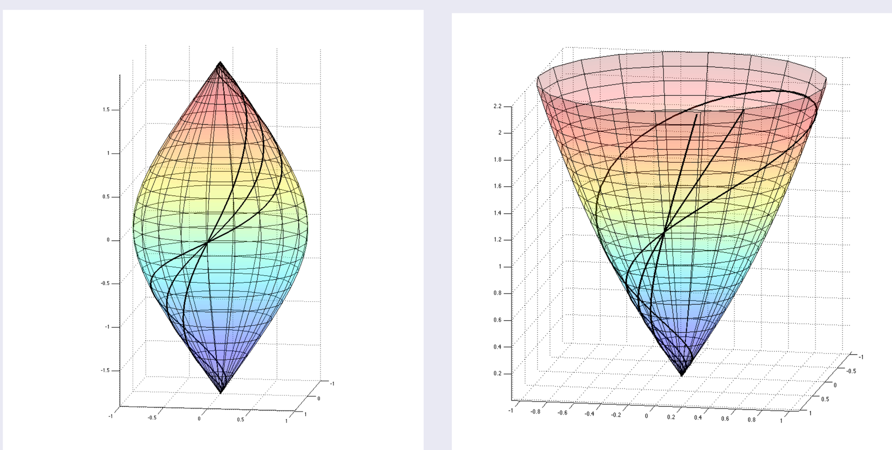
- Lorentz manifolds model spacetime in relativity theory.
- They have one time dimension and one or more spatial dimensions.

Figure: Tangent Space in Lorentz Geometry

- Mathematically defined with metric that is negative definite in one-dimension and positive definite in the spatial dimensions.
- Three types of geodesics: timelike, spacelike, lightlike (null). This property is called causal character.
- We use a signed distance: $+$ for spacelike, $-$ for timelike, 0 for lightlike.

Previous IGL Project

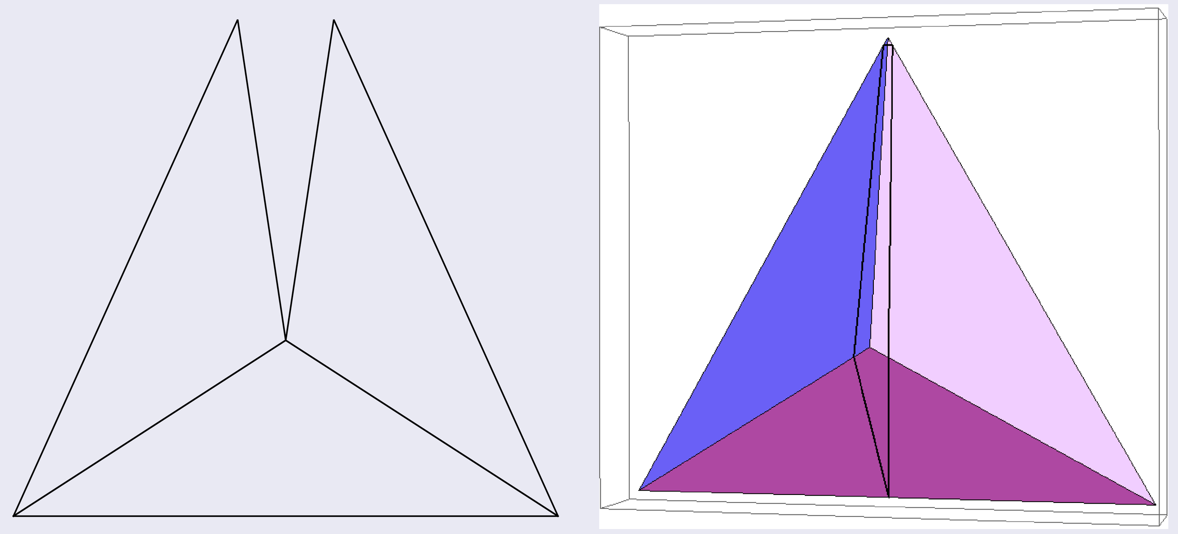
In last semester's IGL project, we created an applet in MATLAB that plots geodesics on Lorentz surfaces of revolution in Minkowski space.



(a) Constant Curvature $+1$ (b) CBB(0) Lorentz Surface

Figure: Geodesics on smooth surfaces of revolution in Minkowski space.

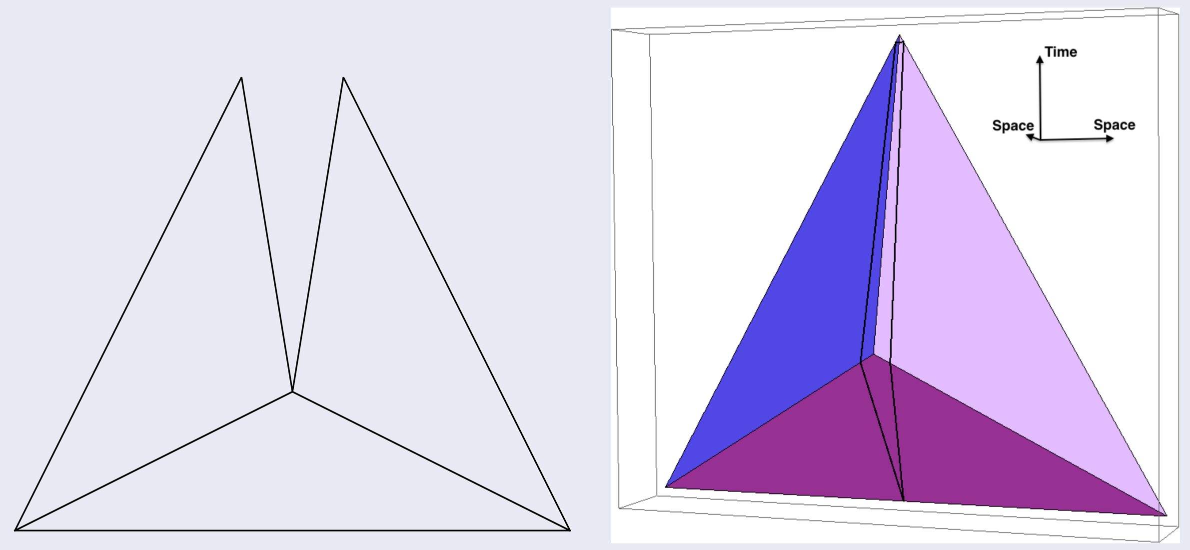
Tetrahedral Surface in Euclidean Space



(a) Unfolded tetrahedral surface in Euclidean plane (b) Tetrahedral surface in Euclidean 3-space

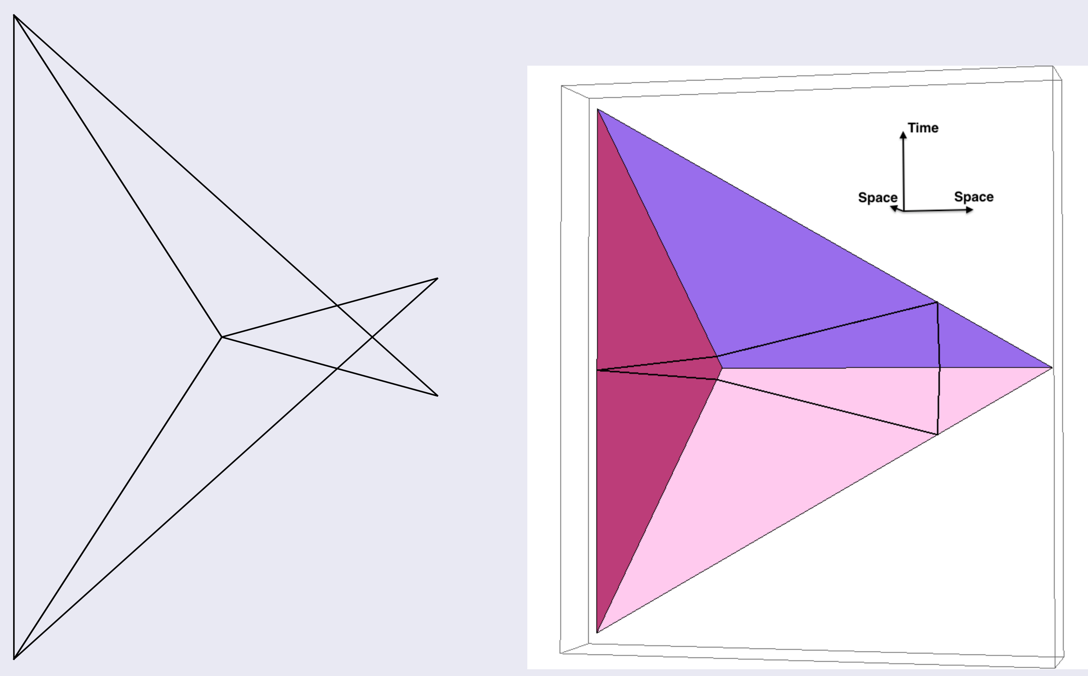
Figure: When unfolded, a curved tetrahedral surface in Euclidean space opens up along the sliced edge. All geodesics come together after passing around the vertex (where the curvature is concentrated).

Tetrahedral Surfaces in Minkowski Space



(a) Unfolded tetrahedral surface in Minkowski plane (b) Tetrahedral surface in Minkowski 3-space

Figure: When unfolded by slicing along a timelike edge, a curved tetrahedral surface in Minkowski space opens up along the sliced edge. Timelike geodesics come together after passing around the vertex (where the curvature is concentrated).



(a) Unfolded tetrahedral surface in Minkowski plane (b) Tetrahedral surface in Minkowski 3-space

Figure: When unfolded by slicing along a spacelike edge, a curved tetrahedral surface in Minkowski space overlaps with itself along the sliced edge. Spacelike geodesics bifurcate around the vertex (where the curvature is concentrated).

IGL Project

- Visualize geodesic triangles on timelike polyhedral surfaces in Minkowski 3-space.
- Geodesics determined by "unfolding" surface along edges.
- z-axis represents time, angle is spatial position.
- Make computer visualizations to aid research in Lorentz spaces with curvature bounds.
- We use Mathematica for the manipulate feature and good visualization.

Mathematics

- Minkowski plane: $\mathbb{E}^2_1 = (\mathbb{R}^2, \langle\cdot,\cdot\rangle)$ with
$$\langle (u_1, u_2), (v_1, v_2)\rangle = u_1 v_1 - u_2 v_2$$
and we give a signed distance by
$$d(u, v) = \operatorname{sgn}(\langle u - v, u - v\rangle)\sqrt{|\langle u - v, u - v\rangle|}.$$
- Similarly, define Minkowski 3-space, $\mathbb{E}^3_1$ with $\langle (u_1, u_2, u_3), (v_1, v_2, v_3)\rangle = u_1 v_1 + u_2 v_2 - u_3 v_3$.
- A path $\gamma : \mathbb{R} \to \mathbb{E}^3_1$ is timelike, lightlike, or spacelike if $\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle < 0$, $= 0$, or > 0, respectively.
- Given a triple of numbers (a, b, c) that does not satisfy the triangle inequality or the reverse triangle inequality, there exists a triangle in $\mathbb{E}^2_1$ with sidelengths a, b, c.
- We obtain geodesics on timelike polyhedral surfaces by "unfolding" surface with Lorentz transformations along edges and drawing straight lines across them.
- The figures in the center illustrate a polyhedral surface formed by three timelike faces of a tetrahedron
- Given a triangle on a polyhedral surface, we can find its comparison triangle in $\mathbb{E}^2_1$.

Lorentz Comparison Geometry

In Alexandrov geometry, a space with non-negative curvature is defined as a space in which triangles are "fatter" than the comparison triangles. Another way to see this is that as two geodesics emanate from a given point, the distance between the end points is a concave function of proper time.

In Lorentz spaces, we can define non-negative curvature in the same fashion, using the convention that timelike distances is negative. Thus,

- geodesics with endpoints connected by spacelike geodesics come together
- geodesics with endpoints connected by timelike geodesics bifurcate

This behavior is seen in the tetrahedral surfaces shown in the center.