The maximum number of s-cliques in graphs without long cycles (or paths)

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- trees
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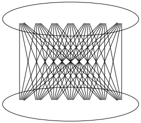

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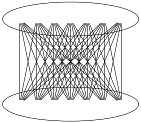

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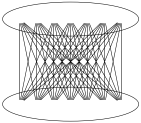

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Theorem (Woodall (inspired by Bondy))

If $e(G) \geq n^2/4 + 1$ then G has a cycle of every length ℓ where $3 \leq \ell \leq 1/2(n+3)$.

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- True for $k = 2, 3, 5$
- $ex(n, C_4) = (\frac{1}{2} + o(1))n^{3/2}$. (Kővari, Sós, Turán 1954, Erdős, Rényi, Sós 1966)

For fixed graphs T and H, define the function

$$ex(n, T, H) =$$

the maximum number of copies of T in an n-vertex H-free graph.
Also define $N_s(G) :=$ the number of copies of K_s in G.

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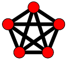
- For $s \le r$, $ex(n, K_s, K_{r+1}) = N_s($  $)$. (Erdős 1962)

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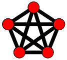
- For $s \leq r$, $ex(n, K_s, K_{r+1}) = N_s($  $)$. (Erdős 1962)
- $ex(n, K_3, C_5) = \Theta(n^{3/2})$. (Bollobás and Győri 2008)

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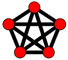
- For $s \leq r$, $ex(n, K_s, K_{r+1}) = N_s($  $)$. (Erdős 1962)
- $ex(n, K_3, C_5) = \Theta(n^{3/2})$. (Bollobás and Győri 2008)
- $ex(n, K_3, C_{2k+1}) \leq \alpha_k ex(n, C_{2k})$. (Alon and Shikhelman 2016, Füredi and Özkahya 2017)

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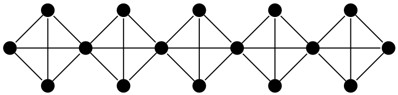
Theorem (Erdős and Gallai 1959)

Let G be an n-vertex graph with no cycle of length at least k. Then $e(G) \leq \frac{n-1}{k-2}\binom{k-1}{2} = \frac{1}{2}(n-1)(k-1)$.



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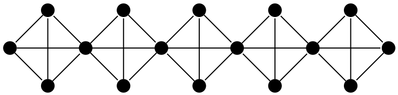
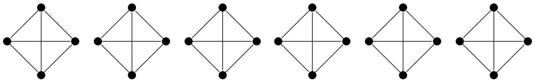
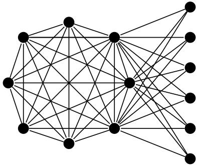


Corollary: $ex(n, P_k) = \frac{n}{k-1}\binom{k-1}{2} = \frac{1}{2}n(k-2)$ where P_k is the path on k vertices.



A refinement of the Erdos–Gallai theorem

Fix $n, k, a \in \mathbb{N}$ such that $2 \leq a \leq \lfloor (k-1)/2 \rfloor$. Define the graph $H_{n,k,a}$:



Define $f(n,k,a) := e(H_{n,k,a}) = \binom{k-a}{2} + a(n-k+a)$.
As a function of a, f is convex and attains its maximum at one of its endpoints, $a = 2$ or $a = \lfloor (k-1)/2 \rfloor$.

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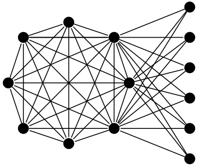

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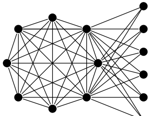
Theorem (Kopylov 1977)

Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then $e(G) \leq \max\{f(n, k, 2), f(n, k, \lfloor (k-1)/2 \rfloor)\}$.

New Results



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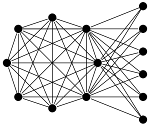

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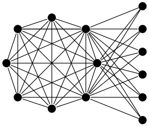

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Theorem (2017)

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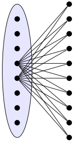

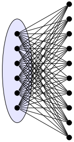

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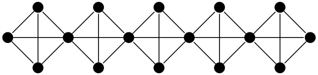

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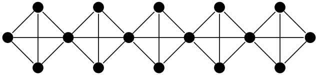

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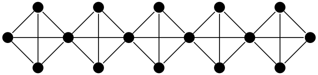

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